On the Lattice of Program Metrics*

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Abstract

In this paper we are concerned with understanding the nature of program metrics for calculi with higher-order types, seen as natural generalizations of program equivalences. Some of the metrics we are interested in are well-known, such as those based on the interpretation of terms in metric spaces and those obtained by generalizing observational equivalence. We also introduce a new one, called the interactive metric, built by applying the well-known Int-Construction to the category of metric complete partial orders. Our aim is then to understand how these metrics relate to each other, i.e., whether and in which cases one such metric refines another, in analogy with corresponding well-studied problems about program equivalences. The results we obtain are twofold. We first show that the metrics of semantic origin, i.e., the denotational and interactive ones, lie *in between* the observational and equational metrics and that in some cases, these inclusions are strict. Then, we give a result about the relationship between the denotational and interactive metrics, revealing that the former is less discriminating than the latter. All our results are given for a linear lambda-calculus, and some of them can be generalized to calculi with graded comonads, in the style of Fuzz.

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1 Introduction

Program equivalence is one of the most important concepts in the semantics of programming languages: every way of giving semantics to programs induces a notion of equivalence, and the various notions of equivalence available for the same language, even when very different from each other, help us understanding the deep nature of the language itself. Indeed, there is not *one* single, preferred way to construct a notion of equivalence for programs. The latter is especially true in presence of higher-order types or in scenarios in which programs have a fundamentally interactive behavior, e.g. in process algebras. For example, the relationship between observational equivalence, the most coarse-grained congruence relation among those which are coherent with the underlying notion of observation, and denotational semantics has led in some cases to so-called full-abstraction results (e.g. [17, 12]), which are known to hold only for *some* denotational models and in *some* programming languages. A similar argument applies to applicative bisimularity, which, e.g., is indeed fully abstract in presence of *probabilistic* effects [8, 9] but not so in presence of *nondeterministic* effects [19].

Equivalences, although central to the theory of programming languages, do not allow us to say anything about all those pairs of programs which, while qualitatively exhibiting different behaviors, behave *similarly* in a quantitative sense. This has led to the study of notions of *distance* between programs, which often take the form of (pseudo-)metrics on the space of programs or their denotations. In this sense we can distinguish at least three defining styles:

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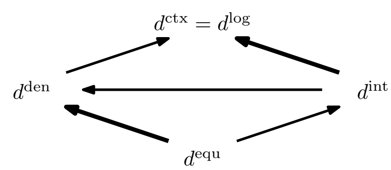


Figure 1 Illustration of our comparison results for program metrics: an arrow $d^{\mathbf{a}} \to d^{\mathbf{b}}$ indicates that $d^{\mathbf{b}}$ is *coarser* (i.e. less discriminating) than $d^{\mathbf{a}}$. Thick arrows indicate *strict* domination.

- First, observational equivalence can be generalized to a metric, maintaining the intrinsic quantification across all contexts, but observing a difference rather than an equality [6, 7].
- There is also an approach obtained by generalizing equational logic, recently introduced by Mardare et al. [21], which has been adapted to higher-order computations with both linear [10] and non-linear [11] types.
- Finally, linear calculi admit a denotational interpretation in the category of metric complete partial orders [3], and this is well-known to work well in presence of graded comonads.

In other words, various definitional styles for program equivalences for higher-order calculi have been proved to have a meaningful metric counterpart, at least when the underlying type system is based on linear or graded types. There is a missing tale in this picture, however, namely the one provided by interactive semantic models akin to game semantics and the geometry of interaction [14], which were key ingredients towards the aforementioned full-abstraction results. Moreover, the relationship between the various notions of distance in the literature has been studied only superficially, and the overall situation is currently less clear than for program equivalences.

The aim of this work is to shed light on the landscape about metrics in higher-programs. Notably, a new metric between programs inspired by Girard's geometry of interaction [14] is defined, being obtained by applying the so-called Int-construction [18, 2] to the category of metric complete partial orders. The result is a denotational model, which, while fundamentally different from existing metric models, provides a natural way to measure the distance between programs, which we will call the *interactive metric*. In the interactive metric, differences between two programs can be observed incrementally, by interacting with the underlying denotational interpretation in the question-answer protocol typical of game semantics and the geometry of interaction.

Technically, the main part of the work is an in-depth study of the relationships between the various metrics existing in the literature, including the interactive metric. Overall, the result of this analysis is the one in Figure 1. The observational metric remains the least discriminating, while the equational metric is proved to be the one assigning the greatest distances to (pairs of) programs. The two metrics of a semantic nature, namely the denotational one and the interactive one, stand in between the two metrics mentioned above, with the interactive metric being more discriminating than the denotational one.

The remainder of this manuscript is structured as follows. After recalling some basic facts about metric spaces in Section 2, in Section 3 we introduce a basic linear programming language over the reals and its associated notion of program metrics; in Section 4 we discuss the logical relation metric and the observational metric; in Section 5 we discuss the equational metric; in Section 6 we introduce the two denotational metrics; Sections 7 and 8 contain our main comparison results, and in Section 9 we discuss the case of graded exponentials.

2 Preliminaries

In this section, we recall the notions of extended pseudo-metric spaces and non-expansive functions. Let $\mathbb{R}^{\infty}_{\geq 0}$ be the set $\{a \in \mathbb{R} \mid a \geq 0\} \cup \{\infty\}$ of non-negative real numbers and infinity. An *extended pseudo-metric space* X consists of a set $|X|$ and a function $d_X : |X| \times |X| \to \mathbb{R}^{\infty}_{\geq 0}$ satisfying the following conditions:

- For all $x \in |X|$, we have $d_X(x, x) = 0$;
- For all $x, y \in |X|$, we have $d_X(x, y) = d_X(y, x)$;
- For all $x, y, z \in |X|$, we have $d_X(x, z) \leq d_X(x, y) + d_X(y, z)$.

In the sequel, we simply refer to extended pseudo-metric spaces as metric spaces, and we denote the underlying set $|X|$ by X.

For metric spaces X and Y, a function $f : X \to Y$ is said to be *non-expansive* when for all $x, x' \in X$, we have $d_Y(fx, fx') \leq d_X(x, x')$. We write **Met** for the category of metric spaces and non-expansive functions. The category **Met** has a symmetric monoidal closed structure $(1, \otimes, \multimap)$ where the metric of the tensor product $X \otimes Y$ is given by

$$d_{X \otimes Y}((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

We suppose that the monoidal product is left associative, and we denote the n-fold monoidal product of X by $X^{\otimes n}$. In the sequel, $\mathbb{R}$ denotes the metric space of real numbers equipped with the absolute distance $d_{\mathbb{R}}(a, b) = |a - b|$.

3 A Linear Programming Language

3.1 Syntax and Operational Semantics

We introduce our target language that is a linear lambda calculus equipped with constant symbols for real numbers and non-expansive functions. We fix a set S of non-expansive functions $f : \mathbb{R}^{\otimes n} \to \mathbb{R}$ with $n \geq 1$. We call n the *arity* of f. For example, S may include addition $+ : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ and trigonometric functions such as $\sin, \cos : \mathbb{R} \to \mathbb{R}$. We assume function symbols $\overline{f}$ for $f \in S$ and constant symbols $\overline{a}$ for real numbers $a \in \mathbb{R}$.

Our language, denoted by Λ_S, is given as follows. Types and environments are given by

$$\text{Types} \quad \tau, \sigma := \mathbf{R} \mid \mathbf{I} \mid \tau \multimap \sigma \mid \tau \otimes \sigma, \qquad \text{Environments} \quad \Gamma, \Delta := \varnothing \mid \Gamma, x : \tau.$$

We denote the set of types by **Ty** and denote the set of environments by **Env**. We always suppose that every variable appears at most once in any environment. For environments Γ and Δ that do not share any variable, we write $\Gamma \# \Delta$ for a *merge* [4, 15] of Γ and Δ, that is an environment obtained by shuffling variables in Γ and Δ preserving the order of variables in Γ and the order of variables in Δ. For example, $(x : \tau, y : \sigma, y' : \sigma', x' : \tau')$ is a merge of $(x : \tau, x' : \tau')$ and $(y : \sigma, y' : \sigma')$. Formally, an environment Ξ is said to be a merge of Γ and Δ when

- Ξ, Γ and Δ are equal to the empty environment; or
- $\Gamma = \Gamma', x : \tau$ and there is a merge Ξ' of Γ' and Δ such that $\Xi = \Xi', x : \tau$; or
- $\Delta = \Delta', x : \tau$ and there is a merge Ξ' of Γ and Δ' such that and $\Xi = \Xi', x : \tau$.

When we write $\Gamma \# \Delta$, we implicitly suppose that no variable is shared by Γ and Δ. Terms,

$$\frac{}{x:\tau \vdash x:\tau} \quad \frac{a \in \mathbb{R}}{\vdash \overline{a} : \mathbf{R}} \quad \frac{}{\vdash * : \mathbf{I}} \quad \frac{f \in S \quad \Gamma_1 \vdash M_1 : \mathbf{R} \quad \dots \quad \Gamma_{\mathrm{ar}(f)} \vdash M_{\mathrm{ar}(f)} : \mathbf{R}}{\Gamma_1 \# \cdots \# \Gamma_{\mathrm{ar}(f)} \vdash \overline{f}(M_1, \dots, M_{\mathrm{ar}(f)}) : \mathbf{R}}$$

$$\frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x:\sigma.\, M : \sigma \multimap \tau} \quad \frac{\Gamma \vdash M : \sigma \multimap \tau \quad \Delta \vdash N : \sigma}{\Gamma \# \Delta \vdash M\, N : \tau} \quad \frac{\Gamma \vdash M : \tau \quad \Delta \vdash N : \sigma}{\Gamma \# \Delta \vdash M \otimes N : \tau \otimes \sigma}$$

$$\frac{\Gamma \vdash M : \mathbf{I} \quad \Delta \vdash N : \tau}{\Gamma \# \Delta \vdash \mathbf{let}\ *\ \mathbf{be}\ M\ \mathbf{in}\ N : \tau} \quad \frac{\Gamma \vdash M : \sigma_1 \otimes \sigma_2 \quad \Delta, x:\sigma_1, y:\sigma_2 \vdash N : \tau}{\Gamma \# \Delta \vdash \mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ N : \tau}$$

Figure 2 Typing Rules

$$\frac{}{V \hookrightarrow V} \quad \frac{M_1 \hookrightarrow \overline{a_1} \quad \dots \quad M_n \hookrightarrow \overline{a_n}}{\overline{f}(M_1, \dots, M_n) \hookrightarrow \overline{f(a_1, \dots, a_n)}} \quad \frac{M \hookrightarrow \lambda x:\tau.\, L \quad N \hookrightarrow V \quad L[V/x] \hookrightarrow U}{M\, N \hookrightarrow U}$$

$$\frac{M \hookrightarrow V \quad N \hookrightarrow U}{M \otimes N \hookrightarrow V \otimes U} \quad \frac{M \hookrightarrow * \quad N \hookrightarrow V}{\mathbf{let}\ *\ \mathbf{be}\ M\ \mathbf{in}\ N \hookrightarrow V} \quad \frac{M \hookrightarrow V \otimes U \quad N[V/x, U/y] \hookrightarrow W}{\mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ N \hookrightarrow W}$$

Figure 3 Evaluation Rules

values and contexts are given by the following BNF.

$$\begin{array}{ll}
\text{Terms} & M, N := x \in \mathbf{Var} \mid \overline{a} \mid * \mid \overline{f}(M_1, \dots, M_{\mathrm{ar}(f)}) \mid M\, N \mid \lambda x:\tau.\, M \mid \\
& \qquad\quad M \otimes N \mid \mathbf{let}\ *\ \mathbf{be}\ M\ \mathbf{in}\ N \mid \mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ N \\
\text{Values} & V, U := \overline{a} \mid * \mid \lambda x:\tau.\, M \mid V \otimes U \\
\text{Contexts} & C[-] := [-] \mid \overline{f}(M, \dots, M', C[-], N', \dots, N) \mid C[-]\, M \mid M\, C[-] \mid \lambda x:\tau.\, C[-] \mid \\
& \qquad\quad C[-] \otimes M \mid M \otimes C[-] \mid \mathbf{let}\ *\ \mathbf{be}\ C[-]\ \mathbf{in}\ M \mid \mathbf{let}\ *\ \mathbf{be}\ M\ \mathbf{in}\ C[-] \mid \\
& \qquad\quad \mathbf{let}\ x \otimes y\ \mathbf{be}\ C[-]\ \mathbf{in}\ M \mid \mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ C[-]
\end{array}$$

Here, a ranges over $\mathbb{R}$, f ranges over S, and x ranges over a countably infinite set $\mathbf{Var}$ of variables. We write $\Gamma \vdash M : \tau$ when the typing judgement is derived from the rules given in Figure 2. Evaluation rules are given in Figure 3. Since Λ_S is a purely linear programming language, for any closed term $\vdash M : \tau$, there is a value $\vdash V : \tau$ such that $M \hookrightarrow V$. For an environment Γ and a type τ, we define $\mathbf{Term}(\Gamma, \tau)$ to be the set of all terms M such that $\Gamma \vdash M : \tau$, and we define $\mathbf{Value}(\tau)$ to be the set of closed values of type τ. We simply write $\mathbf{Term}(\tau)$ for $\mathbf{Term}(\varnothing, \tau)$, that is the set of closed terms of type τ. For a context $C[-]$, we write $C[-] \colon (\Gamma, \tau) \to (\Delta, \sigma)$ when for all terms $\Gamma \vdash M : \tau$, we have $\Delta \vdash C[M] : \sigma$.

We adopt Church-style lambda abstraction so that every type judgement $\Gamma \vdash M : \tau$ has a unique derivation, which makes it easier to define denotational semantics for Λ_S. Except for this point, our language can be understood as a fragment of Fuzz [25]— the typing judgment $x : \sigma, \dots, y : \rho \vdash M : \tau$ corresponds to $x :_1 \sigma, \dots, y :_1 \rho \vdash M : \tau$ in Fuzz. In Section 9, we discuss extending our results in this paper to a richer language, closer to the one from [25].

3.2 Equational Theory

In this paper we consider an equational theory for Λ_S, which will turn out to be instrumental to define a notion of well-behaving family of metrics for Λ_S called admissibility (Section 3.3) and to give a quantitative equational theory for Λ_S (Section 5). In both cases, if two terms are to be considered equal, then the distance between them is required to be 0. Here, we

$$\frac{\Gamma \vdash M : \tau}{\Gamma \vdash M = M : \tau} \qquad \frac{\Gamma \vdash M = N : \tau}{\Gamma \vdash N = M : \tau} \qquad \frac{\Gamma \vdash M = N : \tau \quad \Gamma \vdash N = L : \tau}{\Gamma \vdash M = L : \tau}$$

$$\frac{f(a_1, \ldots, a_n) = b}{\vdash \overline{f}(\overline{a_1}, \ldots, \overline{a_{\mathrm{ar}(f)}}) = \overline{b} : \tau} \qquad \frac{\Gamma \vdash M = N : \tau \quad \Delta \vdash C[M] : \sigma \quad \Delta \vdash C[N] : \sigma}{\Delta \vdash C[M] = C[N] : \sigma}$$

$$\frac{\Gamma, x : \tau \vdash M : \sigma \quad \Delta \vdash N : \tau}{\Gamma \# \Delta \vdash (\lambda x : \tau.\, M)\, N = M[N/x] : \sigma} \qquad \frac{\Gamma \vdash M : \tau \multimap \sigma}{\Gamma \vdash \lambda x : \tau.\, M\, x = M : \tau \multimap \sigma}$$

$$\frac{\Gamma \vdash M : \tau}{\Gamma \vdash \mathbf{let}\ *\ \mathbf{be}\ *\ \mathbf{in}\ M = M : \tau} \qquad \frac{\Gamma \vdash \mathbf{let}\ x \otimes y\ \mathbf{be}\ M \otimes N\ \mathbf{in}\ L : \tau}{\Gamma \vdash \mathbf{let}\ x \otimes y\ \mathbf{be}\ M \otimes N\ \mathbf{in}\ L = L[M/x, N/y] : \tau}$$

$$\frac{\Gamma \vdash M : \mathbf{I}}{\Gamma \vdash \mathbf{let}\ *\ \mathbf{be}\ M\ \mathbf{in}\ * = M : \mathbf{I}} \qquad \frac{\Gamma \vdash M : \tau \otimes \sigma}{\Gamma \vdash \mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ x \otimes y = M : \tau \otimes \sigma}$$

$$\frac{\Gamma \vdash \mathbf{let}\ *\ \mathbf{be}\ M\ \mathbf{in}\ C[N] : \tau}{\Gamma \vdash \mathbf{let}\ *\ \mathbf{be}\ M\ \mathbf{in}\ C[N] = C[\mathbf{let}\ *\ \mathbf{be}\ M\ \mathbf{in}\ N] : \tau}$$

$$\frac{\Gamma \vdash \mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ C[N] : \tau}{\Gamma \vdash \mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ C[N] = C[\mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ N] : \tau}$$

Figure 4 Derivation Rules of Equational Theory for Λ_S

adopt the standard equational theory for the linear lambda calculus [20] extended with the following axiom

$$\frac{f \in S \quad f(a_1, \ldots, a_{\mathrm{ar}(f)}) = b}{\vdash \overline{f}(\overline{a_1}, \ldots, \overline{a_{\mathrm{ar}(f)}}) = \overline{b} : \tau} \ .$$

For terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we write $\Gamma \vdash M = N : \tau$ when the equality is derivable.

We may add some other axioms to the equational theory as long as the axioms are valid when we interpret function symbols $\overline{f}$ as f and constant symbols $\overline{a}$ as a. For example, when $\mathrm{add}\colon \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ is in S, we may add the commutativity law $x : \mathbf{R}, y : \mathbf{R} \vdash \overline{\mathrm{add}}(x, y) = \overline{\mathrm{add}}(y, x) : \mathbf{R}$ to the equational theory. The rest of this paper is not affected by such extensions to the equational theory.

3.3 Admissibility

Let us call a family $\{d_{\Gamma,\tau}\}_{\Gamma \in \mathbf{Env}, \tau \in \mathbf{Ty}}$ in which $d_{\Gamma,\tau}$ is a metric on $\mathbf{Term}(\Gamma, \tau)$ a *metric on* Λ_S. We introduce a class of metrics on Λ_S, which is the object of study of this paper.

▶ **Definition 1** (Admissible Metric). *Let $\{d_{\Gamma,\tau}\}_{\Gamma \in \mathbf{Env}, \tau \in \mathbf{Ty}}$ be a metric on Λ_S. We say that $\{d_{\Gamma,\tau}\}_{\Gamma \in \mathbf{Env}, \tau : \mathbf{Ty}}$* is admissible *when the following conditions hold:*

(A1) For any environment Γ, any type τ, any pair of terms $\Gamma \vdash M : \tau$, $\Gamma \vdash N : \tau$ and any context $C[-]\colon (\Gamma, \tau) \to (\Delta, \sigma)$, we have $d_{\Delta,\sigma}(C[M], C[N]) \leq d_{\Gamma,\tau}(M, N)$.

(A2) For all $a, b \in \mathbb{R}$, we have $d_{\varnothing,\mathbf{R}}(a, b) = |a - b|$.

(A3) For all $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ and all closed values $\vdash V : \tau$ and $\vdash U : \tau$, we have

$$d_{\varnothing, \mathbf{R}^{\otimes n} \otimes \tau}\left(\overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes V, \overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes U\right) \geq |a_1 - b_1| + \cdots + |a_n - b_n|.$$

(A4) If $\Gamma \vdash M = N : \tau$, then $d_{\Gamma,\tau}(M, N) = 0$.

The first condition (A1) states that all contexts are non-expansive, and the second condition (A2) states that the metric on $\mathbf{R}$ coincides with the absolute metric on $\mathbb{R}$. (A3) states that the distance between two terms $\overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes V$ and $\overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes U$ is bounded (from below) by the distance between their "observable fragments" $d_{\mathbb{R}^{\otimes n}}((a_1, \ldots, a_n), (b_1, \ldots, b_n))$. The last condition (A4) states that $d_{\Gamma,\tau}$ subsumes the equational theory for Λ_S.

The definition of admissibility is motivated by the study of Fuzz [25], which is a linear type system for verifying differential privacy [5]. There, Reed and Pierce introduce a syntactically defined metrics on Fuzz using a family of relations called metric relations, and they prove that all programs are non-expansive with respect to the syntactic metric (Theorem 6.4 in [25]). (A1) is motivated by this result. Furthermore, in the definition of the metric relation, the tensor product of types is interpreted as the monoidal product of metric spaces, and the type of real numbers is interpreted as $\mathbb{R}$ with the absolute distance. (A2) and (A3) are motivated by these definitions. In fact, given an admissible metric $\{d_{\Gamma,\tau}\}_{\Gamma \in \mathbf{Env}, \tau \in \mathbf{Ty}}$ on Λ_S, we can show that $d_{\varnothing, \mathbf{R}^{\otimes n}}$ coincides with the metric of $\mathbb{R}^{\otimes n}$.

▶ **Lemma 2.** *If a metric $\{d_{\Gamma,\tau}\}_{\Gamma \in \mathbf{Env}, \tau \in \mathbf{Ty}}$ is admissible, then for all $a_1, b_1, \ldots, a_n, b_n \in \mathbb{R}$,*

$$d_{\varnothing, \mathbf{R}^{\otimes n}}(\overline{a_1} \otimes \cdots \otimes \overline{a_n}, \overline{b_1} \otimes \cdots \otimes \overline{b_n}) = |a_1 - b_1| + \cdots + |a_n - b_n|. \tag{1}$$

Proof. By (A1) and (A3),

$$\begin{aligned}\sum_{1 \le i \le n} |a_i - b_i| &\le d_{\varnothing, \mathbf{R}^{\otimes n} \otimes \mathbf{I}} \left(\overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes *, \overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes *\right) \\ &\le d_{\varnothing, \mathbf{R}^{\otimes n}} \left(\overline{a_1} \otimes \cdots \otimes \overline{a_n}, \overline{b_1} \otimes \cdots \otimes \overline{b_n}\right).\end{aligned}$$

The other inequality follows from (A1), (A2) and triangle inequalities:

$$\begin{aligned}\sum_{1 \le i \le n} |\overline{a_i} - \overline{b_i}| &\ge d_{\varnothing, \mathbf{R}^{\otimes n}} \left(\overline{a_1} \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_n}, \overline{b_1} \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_n}\right) + \sum_{2 \le i \le n} |\overline{a_i} - \overline{b_i}| \\ &\ge d_{\varnothing, \mathbf{R}^{\otimes n}} \left(\overline{a_1} \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_n}, \overline{b_1} \otimes \overline{b_2} \otimes \overline{a_3} \otimes \cdots \otimes \overline{a_n}\right) + \sum_{3 \le i \le n} |\overline{a_i} - \overline{b_i}| \\ &\ge \cdots \\ &\ge d_{\varnothing, \mathbf{R}^{\otimes n}} \left(\overline{a_1} \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_n}, \overline{b_1} \otimes \overline{b_2} \otimes \cdots \otimes \overline{b_n}\right).\end{aligned}$$

◀

The reason that we do not take (1) as the third condition of admissibility and instead rely on the stronger condition (A3) above is that requiring (1) would not allow us to characterize the observational metric (Section 4.2) as the least admissible metric on Λ_S.

4 Logical Metric and Observational Metric

We give two syntactically defined metrics on Λ_S: one is based on logical relations, and the other is given in the style of Morris observational equivalence [24]. We then show that the two metrics coincide. This can be seen as a metric variant of Milner's context lemma [22].

4.1 Logical Metric

The first metric on Λ_S is given by means of a quantitative form of logical relations [25] called *metric logical relations*. Here, we directly define metric logical relations, and then, we define the induced metric on Λ_S. The metric logical relations

$$\{(-) \simeq_r^\tau (-) \subseteq \mathbf{Term}(\tau) \times \mathbf{Term}(\tau)\}_{\tau \in \mathbf{Ty}, r \in \mathbb{R}_{\ge 0}^\infty}$$

are given by induction on τ as follows.

$$\begin{aligned}
M \simeq_r^{\mathbf{R}} N &\iff M \hookrightarrow \overline{a} \text{ and } N \hookrightarrow \overline{b} \text{ and } |a-b| \le r\\
M \simeq_r^{\mathbf{I}} N &\iff M \hookrightarrow * \text{ and } N \hookrightarrow *\\
M \simeq_r^{\tau\otimes\sigma} N &\iff M \hookrightarrow V \otimes V' \text{ and } N \hookrightarrow U \otimes U' \text{ and}\\
&\qquad \exists s, s' \in \mathbb{R}_{\ge 0}^{\infty},\ V \simeq_s^{\tau} U \text{ and } V' \simeq_{s'}^{\sigma} U' \text{ and } s + s' \le r\\
M \simeq_r^{\tau\multimap\sigma} N &\iff M \hookrightarrow \lambda x:\tau.\, M' \text{ and } N \hookrightarrow \lambda x:\tau.\, N' \text{ and}\\
&\qquad \forall V, U \in \mathbf{Value}(\tau), \text{ if } V \simeq_s^{\tau} U,\ \text{then } M'[V/x] \simeq_{r+s}^{\sigma} N'[U/x]
\end{aligned}$$

Then for an environment $\Gamma = (x:\sigma, \dots, y:\rho)$ and a pair of terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we define $d^{\log}_{\Gamma,\tau}(M,N) \in \mathbb{R}_{\ge 0}^{\infty}$ by

$$d^{\log}_{\Gamma,\tau}(M,N) = \inf\{r \in \mathbb{R}_{\ge 0}^{\infty} \mid \lambda x:\sigma.\cdots\lambda y:\rho.\, M \simeq_r^{\sigma\multimap\cdots\multimap\rho\multimap\tau} \lambda x:\sigma.\cdots\lambda y:\rho.\, N\}.$$

▶ **Proposition 3.** *For any environment Γ and any type τ, the function $d^{\log}_{\Gamma,\tau}$ is a metric on $\mathbf{Term}(\Gamma,\tau)$. Furthermore, $\{d^{\log}_{\Gamma,\tau}\}_{\Gamma\in\mathbf{Env},\tau\in\mathbf{Ty}}$ is admissible.*

Proof. It is straightforward to show that d^{obs} given in the next section is a metric on Λ_S and satisfies (A1). Hence, it follows from Theorem 9 that $d^{\log}_{\Gamma,\tau}$ is a metric on $\mathbf{Term}(\Gamma,\tau)$ and satisfies (A1). (A2) and (A3) follow from the definition of $d^{\log}$. The proof of (A4) is given in Corollary 19. ◀

We call $d^{\log}$ *logical metric.*

▶ **Example 1.** For $a \in \mathbb{R}$, we define a term M_a to be

$$\vdash \overline{a} \otimes \overline{a} \otimes V : \mathbf{R} \otimes \mathbf{R} \otimes ((\mathbf{R} \otimes \mathbf{R} \multimap \mathbf{R}) \multimap \mathbf{R}) \qquad \text{where } V = \lambda k : \mathbf{R} \otimes \mathbf{R} \multimap \mathbf{R}.\, k\,\overline{0}\,\overline{0}.$$

Since $d^{\log}_{\varnothing,\mathbf{R}}(\overline{0},\overline{1}) = 1$, we obtain $d^{\log}_{\varnothing,\mathbf{R}\otimes\mathbf{R}\otimes((\mathbf{R}\otimes\mathbf{R}\multimap\mathbf{R})\multimap\mathbf{R})}(M_0, M_1) = 1 + 1 + 0 = 2.$ □

4.2 Observational Metric

We next give a metric, which we call the *observational metric*, that measures distances between terms by observing concrete values produced by any possible context. For terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we define $d^{\mathrm{obs}}_{\Gamma,\tau}(M,N) \in \mathbb{R}_{\ge 0}^{\infty}$ by

$$d^{\mathrm{obs}}_{\Gamma,\tau}(M,N) = \sup_{(n,\sigma,C[-])\in\mathcal{K}(\Gamma,\tau)} \left\{ |a_1 - b_1| + \cdots + |a_n - b_n| \;\middle|\; \begin{array}{l} C[M] \hookrightarrow \overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes V \\ \text{and } C[N] \hookrightarrow \overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes U \end{array} \right\}$$

where $(n,\sigma,C[-]) \in \mathcal{K}(\Gamma,\tau)$ if and only if $C[-]$ is a context from (Γ,τ) to $(\varnothing, \mathbf{R}^{\otimes n} \otimes \sigma)$.

▶ **Example 2.** We consider the term $\vdash M_a : \tau$ given in Example 1 again. By observing M_0 and M_1 by the trivial context $[-]$, we can directly check that $d^{\mathrm{obs}}_{\varnothing,\mathbf{R}\otimes\mathbf{R}\otimes((\mathbf{R}\otimes\mathbf{R}\multimap\mathbf{R})\multimap\mathbf{R})}(M_0, M_1) \ge 2$. (In fact, it follows from Theorem 9 that the distance is equal to 2.) The purpose of the auxiliary type σ in the definition of $\mathcal{K}(\Gamma,\tau)$ is to enable observations of this type. In this case, while the logical metric distinguishes M_0 from M_1, we can not observationally distinguish M_0 from M_1 by means of observations at types $\mathbf{R}^{\otimes n}$ when $S = \emptyset$. See Proposition 4 for impossibility of observational distinction of these terms at $\mathbf{R}^{\otimes n}$. □

▶ **Proposition 4.** *If $S = \emptyset$, then for any $n \in \mathbb{N}$, there is no context*

$$C[-] \colon (\varnothing, \mathbf{R} \otimes ((\mathbf{R}^{\otimes 2} \multimap \mathbf{R}) \multimap \mathbf{R})) \to (\varnothing, \mathbf{R}^{\otimes n}).$$

Proof. We first show that there is no closed term of type $\mathbf{R}^{\otimes 2} \multimap \mathbf{R}$. To see this, for each type τ, we inductively define $|\tau| \in \mathbf{Z}$ by

$$|\mathbf{R}| = 1, \qquad |\mathbf{I}| = 0, \qquad |\tau \otimes \sigma| = |\tau| + |\sigma|, \qquad |\tau \multimap \sigma| = -|\tau| + |\sigma|.$$

We extend the definition of $| - |$ to environments $\Gamma = (x : \tau, \ldots, y : \sigma)$ by letting $|\Gamma|$ to be $|\tau| + \cdots + |\sigma|$. Then by induction on the derivation of $\Gamma \vdash M : \tau$, we can show that if $S = \emptyset$, then $|\Gamma| \leq |\tau|$. Since $|\mathbf{R}^{\otimes 2} \multimap \mathbf{R}| = -1$, we see that there is no closed term of type $\mathbf{R}^{\otimes 2} \multimap \mathbf{R}$. We next show the statement. Let us suppose that there is a context $C[-] : (\varnothing, \mathbf{R} \otimes ((\mathbf{R}^{\otimes 2} \multimap \mathbf{R}) \multimap \mathbf{R})) \to (\varnothing, \mathbf{R}^{\otimes n})$ for some $n \in \mathbb{N}$, and we derive contradiction. Because Λ_S is normalizing, there is a value V such that $C[\overline{0} \otimes (\lambda f : \mathbf{R}^{\otimes 2} \multimap \mathbf{R}.\, f\,\overline{0}\,\overline{0})] \hookrightarrow V$. As we have observed, there is no closed value $U \in \mathbf{Value}(\mathbf{R}^{\otimes 2} \multimap \mathbf{R})$. Therefor, there is no β-reduction of the form $(\lambda f : (\mathbf{R}^{\otimes 2} \multimap \mathbf{R}).\, f\,\overline{0}\,\overline{0})\, U \hookrightarrow U\,\overline{0}\,\overline{0}$ during the reduction $C[\overline{0} \otimes (\lambda f : (\mathbf{R}^{\otimes 2} \multimap \mathbf{R}).\, f\,\overline{0}\,\overline{0})] \hookrightarrow V$. Hence, $\lambda f : (\mathbf{R}^{\otimes 2} \multimap \mathbf{R}).\, f\,\overline{0}\,\overline{0}$ must be a subterm of V, contradicting $V \in \mathbf{Value}(\mathbf{R}^{\otimes n})$. ◀

4.3 Coincidence of the Logical Metric and the Observational Metric

This section is devoted to prove that the logical metric coincides with the observational metric. For the proof, we introduce another family of quantitative relations, called *metric relations* [25]. We define the metric relations

$$\{(-) \cong^{\tau}_{r} (-) \subseteq \mathbf{Term}(\tau) \times \mathbf{Term}(\tau)\}_{\tau \in \mathbf{Ty}, r \in \mathbb{R}^{\infty}_{\geq 0}}$$

by induction on τ as follows.

$$\begin{aligned}
M \cong^{\mathbf{R}}_{r} N &\iff M \hookrightarrow \overline{a} \text{ and } N \hookrightarrow \overline{b} \text{ and } |a - b| \leq r \\
M \cong^{\mathbf{I}}_{r} N &\iff M \hookrightarrow * \text{ and } N \hookrightarrow * \\
M \cong^{\tau \otimes \sigma}_{r} N &\iff M \hookrightarrow V \otimes V' \text{ and } N \hookrightarrow U \otimes U' \text{ and} \\
&\qquad \exists s, s' \in \mathbb{R}^{\infty}_{\geq 0},\ V \cong^{\tau}_{s} U \text{ and } V' \cong^{\sigma}_{s'} U' \text{ and } s + s' \leq r \\
M \cong^{\tau \multimap \sigma}_{r} N &\iff M \hookrightarrow \lambda x : \tau.\, M' \text{ and } N \hookrightarrow \lambda x : \tau.\, N' \text{ and} \\
&\qquad \forall V \in \mathbf{Value}(\tau),\ M'[V/x] \cong^{\sigma}_{r} N'[V/x]
\end{aligned}$$

The only difference between the definition of $\simeq$ and $\cong$ is in the case of the linear function type.

Let us introduce some notations. For an environment $\Gamma = (x_1 : \tau_1, \ldots, x_n : \tau_n)$, we define $\mathbf{Value}(\Gamma)$ to be $\mathbf{Value}(\tau_1) \times \cdots \times \mathbf{Value}(\tau_n)$. Given $\gamma \in \mathbf{Value}(\Gamma)$ and $\Gamma \vdash M : \tau$, we define $\vdash M\gamma : \tau$ in the obvious way. For $\gamma = (V_1, \ldots, V_n), \delta = (U_1, \ldots, U_n) \in \mathbf{Value}(\Gamma)$, we write $\gamma \simeq^{\Gamma}_{r} \delta$ when there are $s_1, \ldots, s_n \in \mathbb{R}^{\infty}_{\geq 0}$ such that $r \geq s_1 + \cdots + s_n$ and $V_1 \simeq^{\tau_1}_{s_1} U_1, \ldots, V_n \simeq^{\tau_n}_{s_n} U_n$ hold.

▶ **Lemma 5.** *For any environment $\Gamma = (x_1 : \tau_1, \ldots, x_n : \tau_n)$ and any pair of terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, if $\gamma \in \mathbf{Value}(\Gamma)$, then $M\gamma \cong^{\tau}_{d^{\mathrm{obs}}_{\Gamma,\tau}(M,N)} N\gamma$.*

Proof. We prove the statement by induction on τ. (When $\tau = \mathbf{R}$) Let $\gamma = (V_1, \ldots, V_n)$ be an element of $\mathbf{Value}(\Gamma)$. For $a, b \in \mathbb{R}$ such that $M\gamma \hookrightarrow \overline{a}$ and $N\gamma \hookrightarrow \overline{b}$, we show that $|a - b| \leq d^{\mathrm{obs}}_{\Gamma,\mathbf{R}}(M, N)$. Let a context $C[-]$ be

$$(\lambda x_1 : \tau_1. \cdots \lambda x_n : \tau_n. [-] \otimes *)\, V_1 \cdots V_n.$$

Then, since we have $C[M] \hookrightarrow \overline{a} \otimes *$ and $C[N] \hookrightarrow \overline{b} \otimes *$, we obtain $|a - b| \leq d^{\mathrm{obs}}_{\Gamma,\mathbf{R}}(M, N)$. (When $\tau = \sigma_1 \otimes \sigma_2$) Let $\gamma = (V_1, \ldots, V_n)$ be an element of $\mathbf{Value}(\Gamma)$. For $U_1, V_1 \in \mathbf{Value}(\sigma_1)$

and $U_2, V_2 \in \mathbf{Value}(\sigma_2)$ such that $M\gamma \hookrightarrow U_1 \otimes U_2$ and $N\gamma \hookrightarrow V_1 \otimes V_2$, we show that there are $s, s' \in \mathbb{R}^{\infty}_{\geq 0}$ such that $U_1 \cong^{\sigma_1}_{s} W_1$ and $U_2 \cong^{\sigma_2}_{s'} W_2$ and $s + s' \leq d^{\mathrm{obs}}_{\Gamma,\sigma_1\otimes\sigma_2}(M, N)$. By the induction hypothesis on σ_1 and σ_2, we obtain $U_1 \cong^{\sigma_1}_{d^{\mathrm{obs}}_{\varnothing,\sigma_1}(U_1,W_1)} W_1$ and $U_2 \cong^{\sigma_2}_{d^{\mathrm{obs}}_{\varnothing,\sigma_2}(U_2,W_2)} W_2$. Hence, by the definition of $\cong$, we have $M\gamma \cong^{\sigma_1\otimes\sigma_2}_{d^{\mathrm{obs}}_{\varnothing,\sigma_1}(U_1,W_1)+d^{\mathrm{obs}}_{\varnothing,\sigma_2}(U_2,W_2)} N\gamma$. It remains to check that $d^{\mathrm{obs}}_{\varnothing,\sigma_1}(U_1, W_1) + d^{\mathrm{obs}}_{\varnothing,\sigma_2}(U_2, W_2) \leq d^{\mathrm{obs}}_{\Gamma,\sigma_1\otimes\sigma_2}(M, N)$. To see this, we show that for any $t_1 < d^{\mathrm{obs}}_{\varnothing,\sigma_1}(U_1, W_1)$ and $t_2 < d^{\mathrm{obs}}_{\varnothing,\sigma_2}(U_2, W_2)$, we have $t_1 + t_2 \leq d^{\mathrm{obs}}_{\Gamma,\sigma_1\otimes\sigma_2}(M, N)$. Given such t_1 and t_2, we can find contexts $C_1[-] \colon (\varnothing, \sigma_1) \to (\varnothing, \mathbf{R}^{\otimes n} \otimes \rho_1)$ and $C_2[-] \colon (\varnothing, \sigma_2) \to (\varnothing, \mathbf{R}^{\otimes m} \otimes \rho_2)$ such that

$$
\begin{aligned}
C_1[U_1] &\hookrightarrow \overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes W_1,\\
C_1[V_1] &\hookrightarrow \overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes W_1',\\
C_2[U_2] &\hookrightarrow \overline{c_1} \otimes \cdots \otimes \overline{c_m} \otimes W_2,\\
C_2[V_2] &\hookrightarrow \overline{d_1} \otimes \cdots \otimes \overline{d_m} \otimes W_2',
\end{aligned}
$$

and

$$
t_1 \leq |a_1 - b_1| + \cdots + |a_n - b_n|, \qquad t_2 \leq |c_1 - d_1| + \cdots + |c_m - d_m|.
$$

We define a context $D[-]$ by

$$
\begin{aligned}
D[-] = \ &\mathbf{let}\ y \otimes z\ \mathbf{be}\ (\lambda x_1 : \tau_1.\ \cdots \lambda x_n : \tau_n.\, [-])\, V_1\ \cdots\ V_n\ \mathbf{in}\\
&\qquad \mathbf{let}\ v \otimes v'\ \mathbf{be}\ C_1[y]\ \mathbf{in}\ \mathbf{let}\ u \otimes u'\ \mathbf{be}\ C_2[z]\ \mathbf{in}\ (H\,(v \otimes u)) \otimes (v' \otimes u')
\end{aligned}
$$

where y and z are fresh variables that do not appear in $C[-]$ nor $D[-]$, and $\vdash H \colon \mathbf{R}^{\otimes n} \otimes \mathbf{R}^{\otimes m} \to \mathbf{R}^{\otimes(n+m)}$ is a value that changes bracketing by using let-bindings. Then, we obtain

$$
\begin{aligned}
B[M] &\hookrightarrow \overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes \overline{c_1} \otimes \cdots \otimes \overline{c_m} \otimes (W_1 \otimes W_2),\\
B[N] &\hookrightarrow \overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes \overline{d_1} \otimes \cdots \otimes \overline{d_m} \otimes (W_1' \otimes W_2').
\end{aligned}
$$

Hence, $t_1 + t_2 \leq d^{\mathrm{obs}}_{\Gamma,\sigma_1\otimes\sigma_2}(M, N)$. (When $\tau = \sigma_1 \multimap \sigma_2$) Let $\gamma = (V_1, \ldots, V_n)$ be an element of $\mathbf{Value}(\Gamma)$. Given $\lambda x : \sigma_1.\, M'$ and $\lambda x : \sigma_1.\, N'$ in $\mathbf{Value}(\sigma_1 \multimap \sigma_2)$ such that $M\gamma \hookrightarrow \lambda x : \sigma_1.\, M'$ and $N\gamma \hookrightarrow \lambda x : \sigma_1.\, N'$, we show that for any $U \in \mathbf{Value}(\sigma_1)$, we have $M'[U/x] \cong^{\sigma_2}_{d^{\mathrm{obs}}_{\Gamma,\sigma_1\multimap\sigma_2}(M,N)} N'[U/x]$. By the induction hypothesis on σ_2 and the definition of $\cong$, we have

$$
M'[U/x] \cong^{\sigma_2}_{d^{\mathrm{obs}}_{\varnothing,\sigma_2}((\lambda x:\sigma_1.\,M')\,U,(\lambda x:\sigma_1.\,N')\,U)} N'[U/x].
$$

Hence, it remains to check that $d^{\mathrm{obs}}_{\varnothing,\sigma_2}((\lambda x : \sigma_1.\, M')\, U, (\lambda x : \sigma_1.\, N')\, U) \leq d^{\mathrm{obs}}_{\Gamma,\sigma_1\multimap\sigma_2}(M, N)$. To see this, we show that for any $r < d^{\mathrm{obs}}_{\varnothing,\sigma_2}((\lambda x : \sigma_1.\, M')\, U, (\lambda x : \sigma_1.\, N')\, U)$, we have $r \leq d^{\mathrm{obs}}_{\Gamma,\sigma_1\multimap\sigma_2}(M, N)$. Since $r < d^{\mathrm{obs}}_{\varnothing,\sigma_2}((\lambda x : \sigma_1.\, M')\, U, (\lambda x : \sigma_1.\, N')\, U)$, there is a context

$$
C[-] \colon (\varnothing, \sigma_2) \to (\varnothing, \mathbf{R}^{\otimes m} \otimes \upsilon)
$$

such that

$$
\begin{aligned}
C[(\lambda x : \sigma_1.\, M')\, U] &\hookrightarrow \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes V\\
C[(\lambda x : \sigma_1.\, N')\, U] &\hookrightarrow \overline{b_1} \otimes \cdots \otimes \overline{b_m} \otimes W
\end{aligned}
$$

and $r \leq |a_1 - b_1| + \cdots + |a_m - b_m|$. We define $D[-]$ by

$$
D[-] = (\lambda y : \sigma_1 \multimap \sigma_2.\, C[y\, U])\, ((\lambda x_1 : \tau_1.\ \cdots \lambda x_n : \tau_n.\, [-])\, V_1\ \cdots\ V_n).
$$

Since

$$D[M] \hookrightarrow \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes V, \qquad D[N] \hookrightarrow \overline{b_1} \otimes \cdots \otimes \overline{b_m} \otimes W,$$

we see that $r \le d^{\mathrm{obs}}_{\Gamma,\sigma_1 \multimap \sigma_2}(M, N)$. ◀

▶ **Lemma 6.** *Let $\Gamma = (x_1 : \tau_1, \ldots, x_n : \tau_n)$ be an environment, and let $\gamma, \gamma' \in \mathbf{Value}(\Gamma)$ be substitutions. If $\gamma \simeq^{\Gamma}_r \gamma'$, then for any term $\Gamma \vdash M : \tau$, we have $M\gamma \simeq^{\tau}_r M\gamma'$.*

Proof. By induction on the derivation of $\Gamma \vdash M : \tau$. ◀

▶ **Lemma 7.** *Let τ be a type.*
1. *For any $\vdash M : \tau$, we have $M \simeq^{\tau}_0 M$.*
2. *For any $\vdash M, N, L : \tau$, if $M \simeq^{\tau}_r N$ and $N \simeq^{\tau}_s L$, then $M \simeq^{\tau}_{r+s} L$.*
3. *For any $\vdash M : \tau$, we have $M \cong^{\tau}_0 M$.*
4. *For any $\vdash M, N, L : \tau$, if $M \cong^{\tau}_r N$ and $N \cong^{\tau}_s L$, then $M \cong^{\tau}_{r+s} L$.*

Proof. (1) follows from Lemma 6. (2) By induction on τ. For the case of $\tau = \tau_1 \multimap \tau_2$, we use (1). (3 and 4) By induction on τ. ◀

▶ **Lemma 8.** *For any type τ and any $r \in \mathbb{R}^{\infty}_{\ge 0}$, $M \simeq^{\tau}_r N$ if and only if $M \cong^{\tau}_r N$.*

Proof. By induction on τ. The only non-trivial case is $\tau = \tau_1 \multimap \tau_2$. We first show that $M \simeq^{\tau}_r N$ implies $M \cong^{\tau}_r N$. Let $\lambda x : \tau_1. M'$ and $\lambda x : \tau_1. N'$ be values such that $M \hookrightarrow \lambda x : \tau_1. M'$ and $N \hookrightarrow \lambda x : \tau_1. N'$. We show that for any $V \in \mathbf{Value}(\tau_1)$, we have $M'[V/x] \cong^{\tau_2}_r N'[V/x]$. Given $V \in \mathbf{Value}(\tau_1)$, by Lemma 7, we have $M'[V/x] \simeq^{\tau_2}_r N'[V/x]$. By the induction hypothesis on τ_2, we obtain the conclusion $M'[V/x] \cong^{\tau_2}_r N'[V/x]$. We next suppose that $M \cong^{\tau}_r N$ and $M \hookrightarrow \lambda x : \tau_1. M'$ and $N \hookrightarrow \lambda x : \tau_1. N'$. For all $V, U \in \mathbf{Value}(\tau_1)$, we show that if $V \simeq^{\tau_1}_s U$, then $M'[V/x] \simeq^{\tau_2}_{r+s} N'[U/x]$. By Lemma 6, we have $M'[V/x] \simeq^{\tau_2}_s M'[U/x]$. From the assumption $M \cong^{\tau}_r N$, we obtain $M'[U/x] \cong^{\tau_2}_r N'[U/x]$. Then it follows from the induction hypothesis on τ_2 that $M'[U/x] \simeq^{\tau_2}_r N'[U/x]$. Finally, it follows from by Lemma 7 that $M'[V/x] \simeq^{\tau_2}_{r+s} N'[U/x]$. ◀

▶ **Theorem 9.** *For any environment Γ and any type τ, we have $d^{\mathrm{obs}}_{\Gamma,\tau} = d^{\log}_{\Gamma,\tau}$.*

Proof. It follows from Lemma 5 and Lemma 8 that $d^{\mathrm{obs}}_{\Gamma,\tau} \ge d^{\log}_{\Gamma,\tau}$. For the other inequality, we show that if $d^{\log}_{\Gamma,\tau}(M, N) \le r$, then $d^{\mathrm{obs}}_{\Gamma,\tau}(M, N) \le r$. For simplicity, we suppose that $\Gamma = (x : \sigma)$, and we define V for $\lambda x : \sigma. M$ and write U for $\lambda x : \sigma. N$. When $d^{\log}_{\Gamma,\tau}(M, N) \le r$, we have $V \simeq^{\sigma \multimap \tau}_r U$. Then, by Lemma 6, for any context $C[-] : (\Gamma, \tau) \to (\varnothing, \mathbf{R}^{\otimes m} \otimes v)$, we have $C[V\,x] \simeq^{\mathbf{R}^{\otimes m} \otimes v}_r C[U\,x]$. From this, it is not difficult to derive $C[M] \simeq^{\mathbf{R}^{\otimes m} \otimes v}_r C[N]$. By the definition of $\simeq$, we obtain $d^{\mathrm{obs}}_{\Gamma,\tau}(M, N) \le r$. ◀

5 Equational Metric

We give another syntactic metric on Λ_S, which we call the *equational metric*. This is essentially the quantitative equational theory from [10] without the rules called **weak**, **join** and **Arch**. We exclude these rules since they do not affect the equational metric d^{equ} given below. See Remark 11 for a proof.

For terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, and for $r \in \mathbb{R}^{\infty}_{\ge 0}$, we write

$$\Gamma \vdash M \approx_r N : \tau$$

$$\frac{\Gamma \vdash M = N : \tau}{\Gamma \vdash M \approx_0 N : \tau} \qquad \frac{\Gamma \vdash M \approx_r N : \tau}{\Gamma \vdash N \approx_r M : \tau} \qquad \frac{\Gamma \vdash M \approx_r N : \tau \quad \Gamma \vdash N \approx_s L : \tau}{\Gamma \vdash M \approx_{r+s} L : \tau}$$

$$\frac{|a-b| \leq r}{\vdash \overline{a} \approx_r \overline{b} : \mathbf{R}} \qquad \frac{\Gamma \vdash M \approx_r N : \tau \quad C[-] : (\Gamma, \tau) \to (\Delta, \sigma)}{\Delta \vdash C[M] \approx_r C[N] : \sigma}$$

Figure 5 Derivation Rules for $\Gamma \vdash M \approx_r N : \tau$

when we can derive the judgement from the rules in Figure 5. Then, for terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$ we define $d^{\mathrm{equ}}_{\Gamma,\tau}(M, N) \in \mathbb{R}^{\infty}_{\geq 0}$ by

$$d^{\mathrm{equ}}_{\Gamma,\tau}(M, N) = \inf\{r \in \mathbb{R}^{\infty}_{\geq 0} \mid \Gamma \vdash M \approx_r N : \tau\}.$$

▶ **Proposition 10.** *For any environment Γ and any type τ, the function $d^{\mathrm{equ}}_{\Gamma,\tau}$ is a metric on $\mathbf{Term}(\Gamma, \tau)$. Furthermore, $\{d^{\mathrm{equ}}_{\Gamma,\tau}\}_{\Gamma \in \mathbf{Env}, \tau \in \mathbf{Ty}}$ is admissible.*

Proof. It is straightforward to check that $d^{\mathrm{equ}}_{\Gamma,\tau}$ is a metric on $\mathbf{Term}(\Gamma, \tau)$. It is also straightforward to check (A1) and (A4). (A2) and (A3) follow from semantic observation (Corollary 19). ◀

▶ **Example 3.** The equational metric measures differences between terms by comparing their subterms. For example, we have $\vdash \overline{2} =_1 \overline{3} : \mathbf{R}$, and therefore, $k : \mathbf{R} \multimap \mathbf{R} \vdash k\,\overline{2} =_1 k\,\overline{3} : \mathbf{R}$ holds. From this, we see that $d^{\mathrm{equ}}_{(k:\mathbf{R}\multimap\mathbf{R}),\mathbf{R}}(k\,\overline{2}, k\,\overline{3}) \leq 1$. In fact, this is an equality. This follows from $d^{\mathrm{obs}}_{(k:\mathbf{R}\multimap\mathbf{R}),\mathbf{R}}(k\,\overline{0}, k\,\overline{1}) \geq 1$, which is easy to check, and Theorem 18 below. □

In general, we have $d^{\mathrm{obs}}_{\Gamma,\tau}(M, N) < d^{\mathrm{equ}}_{\Gamma,\tau}(M, N)$, i.e., the equational metric is strictly more discriminating than the observational metric (Theorem 18), which is proved by semantically inspired metrics in the next section.

▶ Remark 11. The following rules

$$\frac{r \geq s \quad \Gamma \vdash M \approx_s N : \tau}{\Gamma \vdash M \approx_r N : \tau}\,(\mathbf{weak}) \qquad \frac{\Gamma \vdash M \approx_r N : \tau \quad \Gamma \vdash M \approx_s N : \tau}{\Gamma \vdash M \approx_{\min\{r,s\}} N : \tau}\,(\mathbf{join})$$

$$\frac{\forall r > s,\ \Gamma \vdash M \approx_r N : \tau}{\Gamma \vdash M \approx_s N : \tau}\,(\mathbf{Arch})$$

considered in [10] is absent in Figure 5 since they do not affect the equational metric. To see this, let us define $\Gamma \vdash M \approx^{+}_{r} N : \tau$ to be the family of binary relations generated by the rules in Figure 5 with the rules **weak**, **join** and **Arch**. Then we have

$$d^{\mathrm{equ}}_{\Gamma,\tau}(M, N) = \inf\{r \in \mathbb{R}^{\infty}_{\geq 0} \mid \Gamma \vdash M \approx^{+}_{r} N : \tau\}.$$

In fact, since $\Gamma \vdash M \approx_r N : \tau$ implies $\Gamma \vdash M \approx^{+}_{r} N : \tau$ for all $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we have $d^{\mathrm{equ}}_{\Gamma,\tau}(M, N) \geq \inf\{r \in \mathbb{R}^{\infty}_{\geq 0} \mid \Gamma \vdash M \approx^{+}_{r} N : \tau\}$. On the other hand, since the following family of binary relations

$$\Gamma \vdash M \dot{\approx}_r N : \tau \iff d^{\mathrm{equ}}_{\Gamma,\tau}(M, N) \leq r$$

satisfies the rules in Figure 5 and the above three rules, if $\Gamma \vdash M \approx^{+}_{r} N : \tau$, then $d^{\mathrm{equ}}_{\Gamma,\tau}(M, N) \leq r$. Hence, $d^{\mathrm{equ}}_{\Gamma,\tau}(M, N) \leq \inf\{r \in \mathbb{R}^{\infty}_{\geq 0} \mid \Gamma \vdash M \approx^{+}_{r} N : \tau\}$.

6 Models of Λ_S and Associated Metrics

Now, we move our attention to semantically derived metrics on Λ_S. We first give a notion of models of Λ_S based on **Met**-enriched symmetric monoidal closed categories. **Met**-enriched symmetric monoidal closed categories are studied in [10] as models of quantitative equational theories for the linear lambda calculus. Then, we give two examples of semantic metrics on Λ_S: one is based on domain theory, and the other is based on Geometry of Interaction.

6.1 Met-enriched Symmetric Monoidal Closed Category

We say that a symmetric monoidal closed category $(\mathcal{C}, I, \otimes, \multimap)$ is **Met**-*enriched* when every hom-set $\mathcal{C}(X, Y)$ has the structure of a metric space subject to the following conditions:

- the composition is a morphism in **Met** from $\mathcal{C}(X, Y) \otimes \mathcal{C}(Z, X)$ to $\mathcal{C}(Z, Y)$; and
- the tensor is a morphism in **Met** from $\mathcal{C}(X, Y) \otimes \mathcal{C}(Z, W)$ to $\mathcal{C}(X \otimes Z, Y \otimes W)$; and
- the currying operation is an isomorphism in **Met** from $\mathcal{C}(X \otimes Y, Z)$ to $\mathcal{C}(X, Y \multimap Z)$.

For morphisms $f, g \colon X \to Y$ in $\mathcal{C}$, we write $d(f, g)$ for the distance between f and g.

▶ **Definition 12.** *A* pre-model $\mathcal{M} = (\mathcal{C}, I, \otimes, \multimap, \lfloor - \rfloor)$ of Λ_S *is a* **Met**-*enriched symmetric monoidal closed category* $(\mathcal{C}, I, \otimes, \multimap)$ *equipped with an object* $\lfloor \mathbf{R} \rfloor \in \mathcal{C}$ *and families of morphisms* $\{\lfloor a \rfloor \colon I \to \lfloor \mathbf{R} \rfloor\}_{a \in \mathbb{R}}$ *and* $\{\lfloor f \rfloor \colon \lfloor \mathbf{R} \rfloor^{\otimes \mathrm{ar}(f)} \to \lfloor \mathbf{R} \rfloor\}_{f \in S}$.

For a pre-model $\mathcal{M} = (\mathcal{C}, I, \otimes, \multimap, \lfloor - \rfloor)$ of Λ_S, we interpret types as follows:

$$[\![\mathbf{R}]\!]^{\mathcal{M}} = \lfloor \mathbf{R} \rfloor, \quad [\![\mathbf{I}]\!]^{\mathcal{M}} = I, \quad [\![\tau \otimes \sigma]\!]^{\mathcal{M}} = [\![\tau]\!]^{\mathcal{M}} \otimes [\![\sigma]\!]^{\mathcal{M}}, \quad [\![\tau \multimap \sigma]\!]^{\mathcal{M}} = [\![\tau]\!]^{\mathcal{M}} \multimap [\![\sigma]\!]^{\mathcal{M}}.$$

For an environment $\Gamma = (x : \tau, \ldots, y : \sigma)$, we define $[\![\Gamma]\!]^{\mathcal{M}}$ to be $[\![\tau]\!]^{\mathcal{M}} \otimes \cdots \otimes [\![\sigma]\!]^{\mathcal{M}}$. Then, the interpretation $[\![\Gamma \vdash M : \tau]\!]^{\mathcal{M}} \colon [\![\Gamma]\!]^{\mathcal{M}} \to [\![\tau]\!]^{\mathcal{M}}$ in $\mathcal{M}$ is given in the standard manner following [20], and constants are interpreted as follows: $[\![\vdash \overline{a} : \mathbf{R}]\!]^{\mathcal{M}} = \lfloor a \rfloor$,

$$[\![\Gamma \# \cdots \# \Delta \vdash \overline{f}(M, \ldots, N) : \mathbf{R}]\!]^{\mathcal{M}} = \lfloor f \rfloor \circ ([\![M]\!]^{\mathcal{M}} \otimes \cdots \otimes [\![N]\!]^{\mathcal{M}}) \circ \theta$$

where $\theta \colon [\![\Gamma \# \Delta]\!]^{\mathcal{M}} \xrightarrow{\cong} [\![\Gamma]\!]^{\mathcal{M}} \otimes [\![\Delta]\!]^{\mathcal{M}}$ swaps objects following the merge $\Gamma \# \Delta$.

▶ **Definition 13.** *We say that a pre-model* $\mathcal{M} = (\mathcal{C}, I, \otimes, \multimap, \lfloor - \rfloor)$ *of* Λ_S *is a* model *of* Λ_S *if* $\mathcal{M}$ *satisfies the following three conditions.*

- *(M1) For any* $f \in S$*, if* $f(a_1, \ldots, a_{\mathrm{ar}(f)}) = b$*, then* $[\![\overline{f}(\overline{a_1}, \ldots, \overline{a_n})]\!]^{\mathcal{M}} = [\![\overline{b}]\!]^{\mathcal{M}}$.
- *(M2) For all* $a, b \in \mathbb{R}$*,* $d(\lfloor a \rfloor, \lfloor b \rfloor) = |a - b|$.
- *(M3) For all* $x, y \colon I \to X$ *in* $\mathcal{C}$ *and all finite sequences* $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$*, we have*

$$d(\lfloor a_1 \rfloor \otimes \cdots \otimes \lfloor a_n \rfloor \otimes x, \lfloor b_1 \rfloor \otimes \cdots \otimes \lfloor b_n \rfloor \otimes y) \geq |a_1 - b_1| + \cdots + |a_n - b_n|.$$

The first condition corresponds to the reduction rule for function symbols and is necessary to prove soundness for models of Λ_S. The remaining conditions are for admissibility of the metric derived from models of Λ_S.

▶ **Proposition 14** (Soundness). *Let* $\mathcal{M}$ *be a model of* Λ_S*. For any term* $M \in \mathbf{Term}(\tau)$ *and any value* $V \in \mathbf{Value}(\tau)$*, if* $M \hookrightarrow V$*, then* $[\![M]\!]^{\mathcal{M}} = [\![V]\!]^{\mathcal{M}}$.

Proof. By induction on the derivation of $M \hookrightarrow V$. Except for the case $\overline{f}(M_1, \ldots, M_{\mathrm{ar}(f)}) \hookrightarrow \overline{b}$, we can check $[\![M]\!]^{\mathcal{M}} = [\![V]\!]^{\mathcal{M}}$ by using soundness of symmetric monoidal closed categories with respect to the linear lambda calculus [20]. The case $\overline{f}(M_1, \ldots, M_{\mathrm{ar}(f)}) \hookrightarrow \overline{b}$ follows from (M1). ◀

Let $\mathcal{M} = (\mathcal{C}, I, \otimes, \multimap, \lfloor - \rfloor)$ be a model of Λ_S. For an environment Γ and a type τ, we define $d^{\mathcal{M}}_{\Gamma,\tau}$ to be the function

$$d([\![-]\!]^{\mathcal{M}}, [\![-]\!]^{\mathcal{M}})\colon \mathbf{Term}(\Gamma,\tau) \times \mathbf{Term}(\Gamma,\tau) \to \mathbb{R}^{\infty}_{\geq 0}.$$

▶ **Proposition 15.** *For any environment Γ and any type τ, the function $d^{\mathcal{M}}_{\Gamma,\tau}$ is a metric on $\mathbf{Term}(\Gamma,\tau)$. Furthermore, $\{d^{\mathcal{M}}_{\Gamma,\tau}\}_{\Gamma\in\mathbf{Env},\tau\in\mathbf{Ty}}$ is admissible.*

Proof. It follows from **Met**-enrichment that $d^{\mathcal{M}}_{\Gamma,\tau}$ is a metric and (A1) holds. (A2) and (A3) follow from (M2) and (M3). (A4) follows from soundness of symmetric monoidal closed categories with respect to the linear lambda calculus [20] and (M1). ◀

▶ **Example 4.** The symmetric monoidal closed category **Met** of metric spaces and non-expansive functions can be extended to a model $(\mathbf{Met}, I, \otimes, \multimap, \lfloor - \rfloor)$ of Λ_S where we define $\lfloor \mathbf{R} \rfloor \in \mathbf{Met}$ to be $\mathbb{R}$, and for $f \in S$, we define $\lfloor f \rfloor\colon \mathbb{R}^{\otimes \mathrm{ar}(f)} \to \mathbb{R}$ to be f. □

6.2 Denotational Metric

In this section, we recall the notion of metric cpos introduced in [3] as a denotational model of Fuzz, and we give a model of Λ_S based on metric cpos. While we do not need the domain-theoretic feature of metric cpos to model Λ_S, we believe that the category of metric cpos is a good place to explore how metrics from denotational models and metrics from interactive semantic models are related. This is because the domain theoretic structure of the category of metric cpos directly gives rise to an interactive semantic model via Int-construction as we show in Section 6.3.2.

Let us recall the notion of (pointed) metric cpos [3].

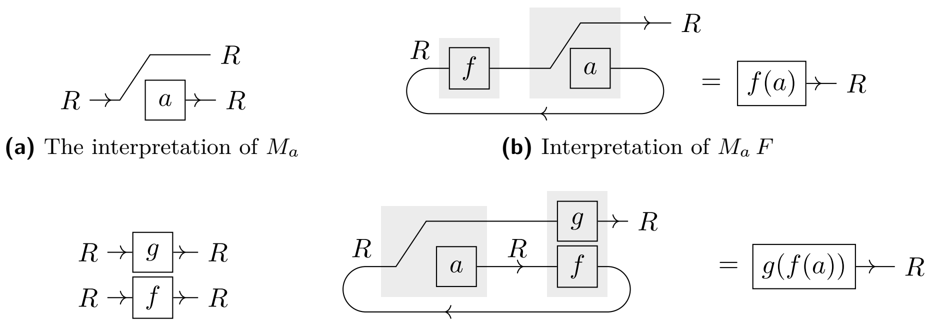
▶ **Definition 16.** *A* (pointed) metric cpo *X consists of a metric space $(|X|, d_X)$ with a partial order $\leq_X$ on $|X|$ such that $(|X|, \leq_X)$ is a (pointed) cpo, and for all ascending chains $(x_i)_{i\in\mathbb{N}}$ and $(x'_i)_{i\in\mathbb{N}}$ on X, we have $d_X\left(\bigvee_{i\in\mathbb{N}} x_i, \bigvee_{i\in\mathbb{N}} x'_i\right) \leq \bigvee_{i\in\mathbb{N}} d_X(x_i, x'_i)$.*

For metric cpos X and Y, a function $f\colon |X| \to |Y|$ is said to be continuous and non-expansive when f is a continuous function from $(|X|, \leq_X)$ to $(|Y|, \leq_Y)$ and is a non-expansive function from $(|X|, d_X)$ to $(|Y|, d_Y)$. Below, we simply write X for the underlying set $|X|$.

Pointed metric cpos and continuous and non-expansive functions form a category, which is denoted by **MetCppo**. The unit object I of **MetCppo** is the unit object of **Met** equipped with the trivial partial order. The tensor product $X \otimes Y$ is given by the tensor product of metric spaces with the componentwise order. The hom-object $X \multimap Y$ is given by the set of continuous and non-expansive functions equipped with the pointwise order and

$$d_{X\multimap Y}(f, g) = \sup_{x\in X} d_Y(fx, gx).$$

We associate **MetCppo** with the structure of a model of Λ_S as follows. We define $\lfloor \mathbf{R} \rfloor$ to be $(\mathbb{R} \cup \{\bot\}, d_R, \leq_R)$ where d_R is the extension of the metric on $\mathbb{R}$ given by $d_R(a, \bot) = \infty$ for all $a \in \mathbb{R}$, and $(\mathbb{R} \cup \{\bot\}, \leq_R)$ is the lifting of the discrete cpo $\mathbb{R}$. For $f \in S$, we define $\lfloor f \rfloor\colon R^{\otimes \mathrm{ar}(f)} \to R$ to be the function satisfying $\lfloor f \rfloor(x_1, \ldots, x_{\mathrm{ar}(f)}) = y \in \mathbb{R}$ if and only if $x_1, \ldots, x_{\mathrm{ar}(f)} \in \mathbb{R}$ and $f(x_1, \ldots, x_{\mathrm{ar}(f)}) = y$. In the sequel, we denote the metric on Λ_S induced by this model by d^{den}, and we call the metric d^{den} the *denotational metric.*



(a) The interpretation of M_a **(b)** Interpretation of $M_a\,F$

(c) Interpretation of N **(d)** Interpretation of $N\,M_a$

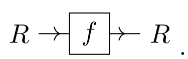
Figure 6 Interpretation of Terms in the Interactive Semantic Model

6.3 Interactive Metric

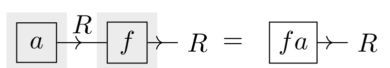
We describe another model of Λ_S, which we call the *interactive semantic model*. In the interactive semantic model, terms are interpreted as strategies interacting with their evaluation environments. Categorically speaking, the construction is based on the notion of *trace operator* and on the related *Int-construction* [18]. Below, we first explain how terms are interpreted, and then, we formally describe the construction of the interactive semantic model.

6.3.1 How Terms are Interpreted, Informally

We present the interpretation of terms in the interactive semantic model using string diagrams without explaining their meaning precisely. We first consider a simple term $F = \lambda x : \mathbf{R}.\, \overline{f}(x)$. Its interpretation is given by the following diagram.



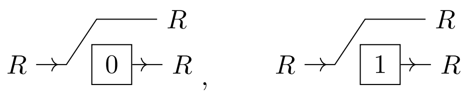
This interpretation means that given an argument $a \in \mathbb{R}$, it returns the evaluation result of $F(\overline{a}) \hookrightarrow \overline{f(a)}$ as follows:



Here, the grey regions denote components corresponding to the argument $\overline{a}$ and the term F. In this example, there is no interaction between functions and their arguments, which instead shows up in higher-order computation. Let us consider $M_a = \lambda k : \mathbf{R} \multimap \mathbf{R}.\, k\,\overline{a}$ for $a \in \mathbb{R}$. The interpretation of this term is given in Figure 6a, and the interpretation of the application $M_a\,F$ is given in Figure 6b, which can be understood as a representation of the following interaction process between the term M_a and its argument F: the term M_a first sends the query $\overline{a}$ to the argument F, and then the argument F invokes $\overline{f}(\overline{a})$. The evaluation result $\overline{f(a)}$ is sent to M_a, and M_a outputs the value $\overline{f(a)}$. We consider another example $N = \lambda k : (\mathbf{R} \multimap \mathbf{R}) \multimap \mathbf{R}.\, \overline{g}(k\,(\lambda x : \mathbf{R}.\, \overline{f}(x)))$. Its interpretation is given in Figure 6c, and the interpretation of $N\,M_a$ is given in Figure 6d. The interaction between N and M_a starts with the query $\overline{a}$ from M_a to N. Then, N invokes $\overline{f}(\overline{a})$. The evaluation result $\overline{f(a)}$ is sent to M_a, and M_a sends $\overline{f(a)}$ to N. Finally, N invokes $\overline{g}(\overline{f(a)})$ and outputs the evaluation result $\overline{g(f(a))}$ of $\overline{g}(\overline{f(a)})$.

In this way, in the interactive semantic model, terms are interpreted as string diagrams that represent "strategies to interact with its arguments". The intuition of interactive metric

d^{int} associated to the interactive semantic model is to measure difference between these strategies. For example, we have $d^{\text{int}}_{\varnothing,(\mathbf{R}\multimap\mathbf{R})\multimap\mathbf{R}}(M_0, M_1) = 1$ since the difference between the two interpretations



are the queries 0 and 1. We note that the interactive semantic model provides an intentional view, and therefore, interactive metric distinguish some observationally equivalent terms. For example, if S has a constant function $c\colon \mathbb{R} \to \mathbb{R}$, then for all $a \in \mathbb{R}$, the terms $L_a = \lambda k : \mathbf{R} \multimap \mathbf{R}.\, \overline{c}(k\,\overline{a})$ are observationally equivalent. On the other hand, we have $d^{\text{int}}_{\varnothing,(\mathbf{R}\multimap\mathbf{R})\multimap\mathbf{R}}(L_0, L_1) = 1$. This is because the interpretation of L_a tells us that for any value $V : \mathbf{R} \multimap \mathbf{R}$, the first event in the evaluation of $L_a\, V$ is to invoke $V\,\overline{a}$.

6.3.2 The Interactive Semantic Model, Formally

In order to formally describe the interactive semantic model, we first observe that the category **MetCppo** has a trace operator, which is necessary to apply the Int-construction to **MetCppo**. For $f\colon X \otimes Z \to Y \otimes Z$ in **MetCppo**, we define $\mathrm{tr}^Z_{X,Y}(f)\colon X \to Y$ by

$$\mathrm{tr}^Z_{X,Y}(f)(x) = \text{the first component of } f(x,z)$$

where z is the least fixed point of the continuous function $f(x,-)\colon Z \to Z$. When we ignore the fragment of metric spaces, the definition of $\mathrm{tr}^Z_{X,Y}(f)$ coincides with the definition of the trace operator associated to the least fixed point operator on the category of pointed cpos and continuous function [15]. Hence, in order to show that $\mathrm{tr}^Z_{X,Y}$ is a trace operator, it is enough to check non-expansiveness of $\mathrm{tr}^Z_{X,Y}(f)$.

▶ **Proposition 17.** *The symmetric monoidal category* $(\mathbf{MetCppo}, I, \otimes)$ *equipped with the family of operators* $\{\mathrm{tr}^Z_{X,Y}\}_{X,Y,Z\in\mathbf{MetCppo}}$ *is a traced symmetric monoidal category.*

Proof. We write $g\colon X \otimes Z \to Z$ and $h\colon X \otimes Z \to Y$ for the continuous and non-expansive functions such that $f(x,a) = (g(x,a), h(x,a))$. To prove non-expansiveness of $\mathrm{tr}^Z_{X,Y}(f)$, we suppose that there are $x, x' \in X$ such that

$$d_Y(\mathrm{tr}^Z_{X,Y}(f)(x), \mathrm{tr}^Z_{X,Y}(f)(x')) > d_X(x,x')$$

and derive a contradiction. By the assumption, $d_X(x,x')$ is finite. We define $a_n, a'_n \in Z$ by

$$a_0 = a'_0 = \bot, \qquad a_{n+1} = g(x, a_n), \qquad a'_{n+1} = g(x', a'_n).$$

We write a_∞ for $\bigvee_{n\in\mathbb{N}} a_n$ and a'_∞ for $\bigvee_{n\in\mathbb{N}} a'_n$. Below, we show that $d_Z(a_\infty, a'_\infty)$ is finite. We first check that $d_Z(a_n, a'_n)$ is finite. The base case is trivial. For the induction step $n > 0$, it follows from non-expansiveness of g that we have

$$d_X(x,x') + d_Z(a_n, a'_n) \geq d_Z(a_{n+1}, a'_{n+1}).$$

Hence, we conclude that $d_Z(a_n, a'_n)$ is finite. We next check that the sequence $d_Z(a_n, a'_n)$ is bounded. Since we have

$$\mathrm{tr}^Z_{X,Y}(f)(x) = h(x, a_\infty) = \bigvee_{n\geq 0} h(x, a_n), \quad \mathrm{tr}^Z_{X,Y}(f)(x') = h(x, a'_\infty) = \bigvee_{n\geq 0} h(x', a'_n),$$

by using Lemma 4.5 in [3], we obtain

$$\liminf_{n\to\infty} d_Y(h(x,a_n),h(x',a'_n)) \geq d_Y(\mathrm{tr}^Z_{X,Y}(f)(x),\mathrm{tr}^Z_{X,Y}(f)(x')) > d_X(x,x').$$

From this, we see that there exists $N \geq 0$ such that for all $n \geq N$,

$$d_Y(h(x,a_n),h(x',a'_n)) > d_X(x,x').$$

Then, it follows from non-expansiveness of f that for all $n \geq N$, we have

$$\begin{aligned} d_X(x,x') + d_Z(a_n,a'_n) &\geq d_Y(h(x,a_n),h(x',a'_n)) + d_Z(a_{n+1},a'_{n+1}) \\ &\geq d_X(x,x') + d_Z(a_{n+1},a'_{n+1}). \end{aligned}$$

Hence, since $d_X(x,x')$ is finite, we have

$$d_Z(a_n,a'_n) \geq d_Z(a_{n+1},a'_{n+1})$$

for all $n \geq N$. Now, we obtain

$$d_Z(a_\infty,a'_\infty) = d_Z\left(\bigvee_{n\geq N} a_n, \bigvee_{n\geq N} a'_n\right) \leq d_X(a_N,a'_N) < \infty.$$

Since

$$d_X(x,x') + d_Z(a_\infty,a'_\infty) \geq d_Y(\mathrm{tr}^Z_{X,Y}(f)(x),\mathrm{tr}^Z_{X,Y}(f)(x')) + d_Z(a_\infty,a'_\infty),$$

we have

$$d_X(x,x') \geq d_Y(\mathrm{tr}^Z_{X,Y}(f)(x),\mathrm{tr}^Z_{X,Y}(f)(x')),$$

which contradicts the assumption. ◀

Now, we can apply the Int-construction to **MetCppo** and obtain a symmetric monoidal closed category **Int**(**MetCppo**). (In fact, what we obtain is a compact closed category, and we only need its symmetric monoidal closed structure to interpret Λ_S.) Objects in **Int**(**MetCppo**) are pairs $X = (X_+, X_-)$ consisting of objects X_+ and X_- in **MetCppo**, and a morphism from X to Y in **Int**(**MetCppo**) is a morphism from $X_+ \otimes Y_-$ to $X_- \otimes Y_+$ in **MetCppo**. The identity on (X_+, X_-) is the symmetry $X_+ \otimes X_- \cong X_- \otimes X_+$, and the composition of $f\colon (X_+,X_-) \to (Y_+,Y_-)$ is given by

$$\mathrm{tr}^{Y_-\otimes Y_+}_{X_+\otimes Z_-, X_-\otimes Z_+}((X_- \otimes \theta)\circ(f\otimes g)\circ(X_+\otimes\theta'))$$

where $\theta\colon Y_+\otimes Y_-\otimes Z_+ \to Z_+\otimes Y_-\otimes Y_+$ and $\theta'\colon Y_-\otimes Y_+\otimes Z_- \to Z_-\otimes Y_-\otimes Y_+$ are the canonical isomorphisms, and we omit some coherence isomorphisms. The symmetric monoidal closed structure of **Int**(**MetCppo**) is given as follows. The tensor unit is (I,I), and the tensor product $X\otimes Y$ is $(X_+\otimes Y_+, X_-\otimes Y_-)$. The hom-object $X \multimap Y$ is $(X_-\otimes Y_+, X_+\otimes Y_-)$. For more details on the categorical structure of **Int**(**MetCppo**), see [18, 26].

We associate **Int**(**MetCppo**) with the structure of a model of Λ_S as follows. We define $\lfloor \mathbf{R} \rfloor$ to be (R,I), and for each $f\in S$, we define $\lfloor f\rfloor\colon (R,I)^{\otimes \mathrm{ar}(f)} \to (R,I)$ by

$$R^{\otimes\mathrm{ar}(f)}\otimes I \xrightarrow{\cong} R^{\otimes \mathrm{ar}(f)} \xrightarrow{\text{the interpretation of } \overline{f} \text{ in } \mathbf{MetCppo}} R \xrightarrow{\cong} I^{\otimes\mathrm{ar}(f)}\otimes R.$$

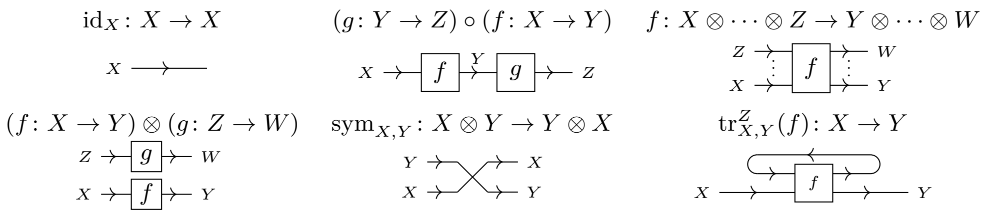


Figure 7 String Diagrams for the Traced Symmetric Monoidal Structure

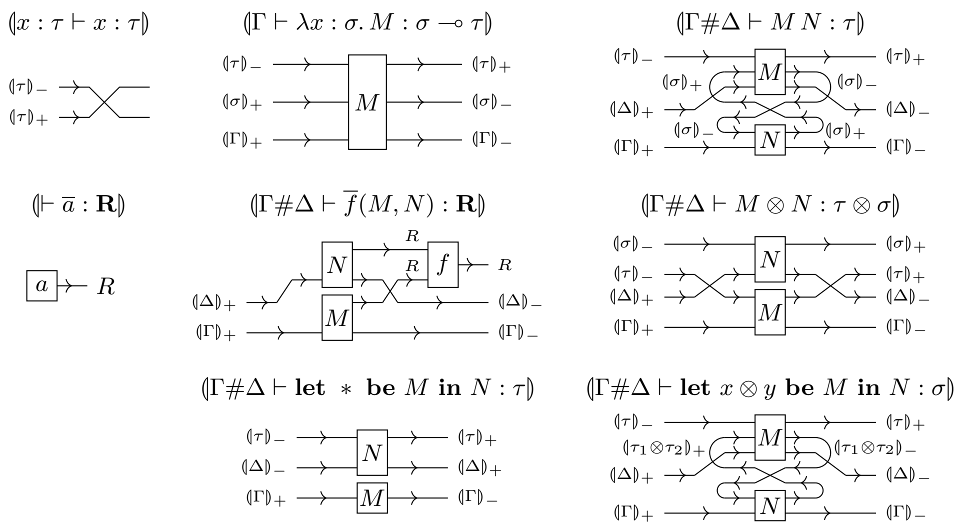


Figure 8 The Interpretation of Λ_S in $\mathbf{Int}(\mathbf{MetCppo})$

We write d^{int} for the metric on Λ_S induced by the interactive semantic model, and we call d^{int} the *interactive metric.*

In Figure 8, we describe the interpretation of Λ_S in $\mathbf{Int}(\mathbf{MetCppo})$ in terms of string diagrams. Here, we write $(\!|\tau|\!)_+$ and $(\!|\tau|\!)_-$ for the positive part and the negative part of the interpretation of τ, and we write $(\!|\Gamma \vdash M : \tau|\!)$ for the interpretation of a term $\Gamma \vdash M : \tau$. See Figure 7 (and [26]) for the meaning of string diagrams. The interpretation $(\!|\Vdash * : \mathbf{I}|\!)$ is not in Figure 8 since $(\!|\Vdash * : \mathbf{I}|\!)$ is the identity on the unit object I, which is presented by zero wires. In the interpretation of $\overline{f}(M_1, \ldots, M_{\mathrm{ar}(f)})$, we suppose that $\mathrm{ar}(f) = 2$ for legibility.

7 Finding Your Way Around the Zoo

We describe how admissible metrics on Λ_S in this paper are related. Below, for metrics $d = \{d_{\Gamma,\tau}\}_{\Gamma \in \mathbf{Env}, \tau \in \mathbf{Ty}}$ and $d' = \{d'_{\Gamma,\tau}\}_{\Gamma \in \mathbf{Env}, \tau \in \mathbf{Ty}}$ on Λ_S, we write $d \le d'$ when for all terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we have $d_{\Gamma,\tau}(M, N) \le d'_{\Gamma,\tau}(M, N)$. We write $d < d'$ when we have $d \le d'$ and $d \ne d'$. Our main results are about the relationships between the various metrics on Λ_S illustrated in Figure 1.

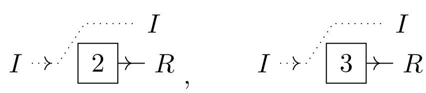
► **Theorem 18.** *The following inclusions hold.*
1. *For any admissible metric d on Λ_S, we have $d^{\mathrm{log}} = d^{\mathrm{obs}} \le d \le d^{\mathrm{equ}}$.*
2. *If a metric d on Λ_S satisfies (A1) and $d^{\mathrm{obs}} \le d \le d^{\mathrm{equ}}$, then d is admissible.*
3. $d^{\mathrm{log}} = d^{\mathrm{obs}} \le d^{\mathrm{den}} < d^{\mathrm{int}} \le d^{\mathrm{equ}}$.

Proof. (Proof of (1)) We first show that $d^{\text{obs}} \leq d$. For any $(n, \sigma, C[-]) \in \mathcal{K}(\Gamma, \tau)$, if $C[M] \hookrightarrow \overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes V$ and $C[N] \hookrightarrow \overline{b_1} \otimes \cdots \otimes \overline{b_n} \otimes U$, then

$$\sum_{1 \leq i \leq n} |a_i - b_i| \overset{\text{(A3)+(A4)}}{\leq} d_{\varnothing, \mathbf{R}^{\otimes n} \otimes \sigma}(C[M], C[N]) \overset{\text{(A1)}}{\leq} d_{\Gamma,\tau}(M, N).$$

By the definition of d^{obs}, we obtain $d^{\text{obs}} \leq d$. We next show that $d \leq d^{\text{equ}}$. We can inductively show that if $\Gamma \vdash M \approx_r N : \tau$, then $d_{\Gamma,\tau}(M, N) \leq r$. In the induction step for $\Gamma \vdash M \approx_0 N : \tau$, we use (A4). In the induction step for $\vdash \overline{a} \approx_r \overline{b} : \mathbf{R}$, we use (A2). In the induction step for $\Delta \vdash C[M] \approx_r C[N] : \sigma$, we use (A1). By the definition of $d^{\text{equ}}_{\Gamma,\tau}(M, N)$, we obtain $d_{\Gamma,\tau}(M, N) \leq d^{\text{equ}}_{\Gamma,\tau}(M, N)$. (Proof of (2)) We check that d satisfies (A2), (A3) and (A4). The condition (A2) holds because $|a - b| = d^{\text{obs}}_{\varnothing,\mathbf{R}}(\overline{a}, \overline{b}) \leq d_{\varnothing,\mathbf{R}}(\overline{a}, \overline{b}) \leq d^{\text{equ}}_{\varnothing,\mathbf{R}}(\overline{a}, \overline{b}) = |a - b|$. (A3) follows from $d^{\text{obs}}_{\Gamma,\tau} \leq d_{\Gamma,\tau}$. (A4) follows from $d_{\Gamma,\tau} \leq d^{\text{equ}}_{\Gamma,\tau}$. (Proof of (3)) The inequalities $d^{\text{obs}} \leq d^{\text{den}}$ and $d^{\text{int}} \leq d^{\text{equ}}$ follow from (3). The proof of the strict inequality $d^{\text{den}} < d^{\text{int}}$ is deferred to the next section. ◀

Concrete metrics in-between d^{obs} and d^{equ} are useful to calculate d^{obs} and d^{equ}. For example, it is not easy to *directly* prove $d^{\text{equ}}_{(k:\mathbf{R} \multimap \mathbf{I}),\mathbf{I}}(k\,\overline{2}, k\,\overline{3}) \geq 1$ since we need to know that *whenever* $k : \mathbf{R} \multimap \mathbf{I} \vdash k\,\overline{2} \approx_r k\,\overline{3} : \mathbf{I}$ is derivable, we have $r \geq 1$. Let us give a semantic proof for the inequality $d^{\text{equ}}_{(k:\mathbf{R} \multimap \mathbf{I}),\mathbf{I}}(k\,\overline{2}, k\,\overline{3}) \geq 1$. Here, we use the interactive semantic model. The interpretations of these terms in the interactive semantic model are



where we can *directly* see the values applied to k. Hence, we obtain $d^{\text{int}}_{(k:\mathbf{R} \multimap \mathbf{I}),\mathbf{I}}(k\,\overline{2}, k\,\overline{3}) = 1$. Then, the claim follows from $d^{\text{int}} \leq d^{\text{equ}}$.

By applying Theorem 18 to d^{den}, we can show admissibility of d^{obs} and d^{equ}.

▶ **Corollary 19.** *The metrics $d^{\text{log}} = d^{\text{obs}}$ and d^{equ} on Λ_S are admissible.*

Proof. We first show admissibility of d^{obs}. As we mentioned in the proof of Proposition 3, it remains to check that d^{obs} satisfies (A4). When $\Gamma \vdash M = N : \tau$, then by Theorem 18, we obtain $d^{\text{obs}}_{\Gamma,\tau}(M, N) \leq d^{\text{den}}_{\Gamma,\tau}(M, N) = 0$. Hence, $d^{\text{log}}_{\Gamma,\tau}(M, N) = d^{\text{obs}}_{\Gamma,\tau}(M, N) = 0$. We next show admissibility of d^{equ}. As we mentioned in the proof of Proposition 10, it remains to check that d^{equ} satisfies (A2) and (A3). Since for all $a, b \in \mathbb{R}$, we have

$$|a - b| \leq d^{\text{den}}_{\varnothing,\mathbf{R}}(\overline{a}, \overline{b}) \leq d^{\text{equ}}_{\varnothing,\mathbf{R}}(\overline{a}, \overline{b}).$$

Hence, d^{equ} satisfies (A2). (A3) can be checked in the same way. ◀

As for semantic metrics, we have the following separation results.

▶ **Proposition 20.** *If $S = \emptyset$, then we have $d^{\text{obs}} < d^{\text{den}}$ and $d^{\text{obs}} < d^{\text{int}}$.*

Proof. We only check the statement $d^{\text{obs}} < d^{\text{den}}$. The other strict inequality can be checked in the same way. Since $d^{\text{obs}} \leq d^{\text{den}}$, we only need to check they are different. Let Γ be $(f : \mathbf{R}^{\otimes 2} \multimap \mathbf{R})$. Then, we have

$$d^{\text{den}}_{\Gamma,\mathbf{R}^{\otimes 2}}(\overline{0} \otimes (f\,\overline{0}\,\overline{0}), \overline{1} \otimes (f\,\overline{0}\,\overline{0})) = 1.$$

On the other hand, as we observed in the proof of Proposition 4, there is no closed term of type $\mathbf{R}^{\otimes 2} \multimap \mathbf{R}$. Hence, $d^{\text{obs}}_{\Gamma,\mathbf{R}^{\otimes 2}}(\overline{0} \otimes (f\,\overline{0}\,\overline{0}), \overline{1} \otimes (f\,\overline{0}\,\overline{0})) = d^{\text{log}}_{\Gamma,\mathbf{R}^{\otimes 2}}(\overline{0} \otimes (f\,\overline{0}\,\overline{0}), \overline{1} \otimes (f\,\overline{0}\,\overline{0})) = 0$. ◀

▶ **Proposition 21.** *We have* $d^{\mathrm{den}}_{(k:\mathbf{R}\multimap\mathbf{I}),\mathbf{I}}(k\,\overline{0}, k\,\overline{1}) = 0$ *and* $d^{\mathrm{int}}_{(k:\mathbf{R}\multimap\mathbf{I}),\mathbf{I}}(k\,\overline{0}, k\,\overline{1}) = 1$*. In particular,* $d^{\mathrm{int}} \not\leq d^{\mathrm{den}}$.

Proof. We have $d^{\mathrm{den}}_{(k:\mathbf{R}\multimap\mathbf{I}),\mathbf{I}}(k\,\overline{0}, k\,\overline{1}) = \sup_{k:\,\lfloor\mathbf{R}\rfloor\to I} d(k(0), k(1)) = 0$. On the other hand, for $a \in \mathbb{R}$, we have $(\!| k\,\overline{a} |\!) = a \colon I \to R$. Hence, $d^{\mathrm{int}}_{(k:\mathbf{R}\multimap\mathbf{I}),\mathbf{I}}(k\,\overline{0}, k\,\overline{1}) = 1$. ◀

8 Comparing the Two Denotational Viewpoints

In this section we show that, by passing from **MetCppo** to the interactive model via the Int-construction, one obtains a more discriminative metric. In other words, our goal is to establish that $d^{\mathrm{den}} < d^{\mathrm{int}}$.

In this section, beyond the evaluation relation defined in Section 3, we will make reference to the standard β-reduction and β-equivalence relations on Λ_S. Indeed, the two semantics we are considering behave differently with respect to these relations: for β-equivalent terms M, N, while their interpretations in **MetCppo** coincide (and thus $d^{\mathrm{den}}(M, N) = 0$), this needs not be the case in the interactive model.

Let us start by making the interactive metric more explicit. Notably, in the case of β-normal terms, computing distances in **Int**(**MetCppo**) can be reduced to computing distances in **MetCppo** as follows: a morphism from Γ to σ in **Int**(**MetCppo**) is a morphism in **MetCppo** from $(\!|\Gamma|\!)_+ \otimes (\!|\sigma|\!)_+$ to $(\!|\Gamma|\!)_- \otimes (\!|\sigma|\!)_+$, where these two objects correspond to tensors of the form $\mathbf{U} \otimes \cdots \otimes \mathbf{U}$, with $\mathbf{U} \in \{\mathbf{I}, \mathbf{R}\}$, More precisely, with any list of types Γ one can associate two natural numbers Γ^+, Γ^- defined inductively by $(\emptyset)^+ = (\emptyset)^- = 0$, $(\mathbf{U}*\Gamma)^+ = 1+\Gamma^+$, $(\mathbf{U}*\Gamma)^- = \Gamma^-$, $(\sigma \multimap \tau*\Gamma)^+ = \sigma^- + \tau^+ + \Gamma^+$, $(\sigma \multimap \tau*\Gamma)^- = \sigma^+ + \tau^- + \Gamma^-$, $(\sigma \otimes \tau * \Gamma)^+ = \sigma^+ + \tau^+ + \Gamma^+$, $(\sigma \otimes \tau * \Gamma)^- = \sigma^- + \tau^- + \Gamma^-$. Then one has the following:

▶ **Proposition 22** (first-order int-terms)**.** *Let M, N be β-normal terms such that $\Gamma \vdash M, N : \sigma$ and let $m = \Gamma^+ + \sigma^-$, $n = \Gamma^- + \sigma^+$. Then there exist* first-order *linear terms $H^M_1, \ldots, H^M_n$, depending on variables $x_1, \ldots, x_m$, and a partition $I_1, \ldots, I_m$ of $\{1, \ldots, m\}$ such that:*

- $\Gamma_j \vdash H^M_j : \mathbf{U}$*, for all $j = 1, \ldots, n$, where $\Gamma_j = \{x_l : \mathbf{U} \mid l \in I_j\}$, with $\mathbf{U} \in \{\mathbf{I}, \mathbf{R}\}$;*
- $[\![M]\!]^{\mathbf{Int}(\mathbf{MetCppo})} = \bigotimes_j [\![H^M_j]\!]^{\mathbf{MetCppo}}$.

Proof.
- if $M = x$, then $\Gamma = \{x : \sigma\}$, so $m = \sigma^+ + \sigma^-$ and $n = \sigma^- + \sigma^+$, hence the variables $\alpha_1, \ldots, \alpha_m$ can be split as $\beta_1, \ldots, \beta_{\sigma^+}, \gamma_1, \ldots, \gamma_{\sigma^-}$, and we let, for $i \leq \sigma^-$, $H^M_i = \gamma_i$, and for $i \geq \sigma^+$, $H^M_{\sigma^- + i} = \alpha_i$;
- if $M = \star$, then $\Gamma = \emptyset$ and $n = 1$, and we let $H^M_1 = \star$;
- if $M = \overline{a}$, then $\Gamma = \emptyset$ and $n = 1$, and we let $H^M_1 = \overline{a}$;
- if $M = \overline{f}(M_1, \ldots, M_k)$, then there is a partition $J_1, \ldots, J_k$ of $1, \ldots, m$, so that $\Gamma_l \vdash M_l : \mathbf{U}$, where Γ_l only contains the variables α_r with $r \in J_l$. Moreover, we have that $m = \Gamma^+ + \mathbf{U}^- = \Gamma^+ = \sum_l (\Gamma_l)^+$ and $n = \Gamma^- + \mathbf{R}^+ = \sum_l (\Gamma_l)^- + 1$. We thus define H^M_i as follows:
  - if $i = \sum_{l=1}^m \Gamma^-_l + j$, with $m < k$ and $j \leq \Gamma^-_{m+1}$, then $H^M_i = H^{M_{m+1}}_j$;
  - if $i = \sum_l^j \Gamma^-_l + 1$, then $H^M_i = \overline{f}(H^{M_1}_{\Gamma^-_1 + 1}, \ldots, H^{M_k}_{\Gamma^-_k + 1})$.
- if $M = \lambda x.M'$, then the H^M_i are defined like the $H^{M'}_i$.
- if $M = xM_1 \ldots M_k$, then there is a partition $J_1, \ldots, J_k$ of $\rho^+ + 1, \ldots, m$ such that $\Gamma = \{x : \rho\} + \sum_{l=1} \Gamma_l$, with Γ_l containing only the variables α_s, for $s \in J_l$, and where $\rho = \sigma_1 \multimap \cdots \multimap \sigma_k \multimap \sigma$ and $\Gamma_l \vdash M_l : \sigma_l$. Then $m = \rho^+ + \sum_l \Gamma^+_l + \sigma^- = \sum_l \sigma^-_l + \sigma^+ + \sum_l \Gamma^+_l + \sigma^-$, so the variables $\alpha_1, \ldots, \alpha_m$ can be identified with the variables occurring in all the terms $H^{M_l}_l$ plus new variables β_s for any negative occurrence in σ and γ_r for any

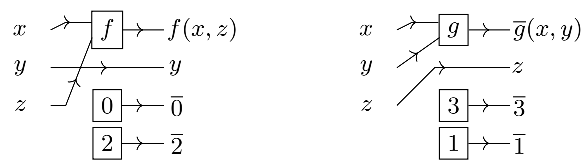


Figure 9 String diagrams with int-terms for $M = \overline{f}(x(y\overline{0}), z\overline{2})$ and $N = \overline{g}(x(z\overline{1}), y\overline{3})$.

positive occurrence in σ; moreover, $n = \rho^- + \sum_l \Gamma_l^- + \sigma^+ = \sum_l \sigma_l^+ + \sigma^- + \sum_l \Gamma_l^- + \sigma^+$. So we define the terms H_i^M as follows:

- for $i = \sum_{l=1}^m \sigma_l^+ + j$ for $m < k$ and $j \leq \sigma_{m+1}^+$, $H_i^M = H_{\Gamma_{m+1}^- + j}^{M_{m+1}}$;
- for $i = \sum_l \sigma_l^+ + s$, for $s \leq \sigma^-$, $H_i^M = \beta_s$;
- for $i = \sum_l \sigma_l^+ + \sigma^- + \sum_{l=1}^m \Gamma_l^- + j$, for $m < k$ and $j \leq \Gamma_{m+1}^-$, $H_i^M = H_j^{M_{m-1}}$;
- for $i = \sum_l \sigma_l^+ + \sigma^- + \sum_l \Gamma_l^- + r$, for $r \leq \sigma^+$, $H_i^M = \gamma_r$.

- if $M = M_1 \otimes M_2$, then $\sigma = \sigma_1 \otimes \sigma_2$ and Γ splits as $\Gamma_1 + \Gamma_2$, with $\Gamma_1 \vdash M_1 : \sigma_1$ and $\Gamma_2 \vdash M_2 : \sigma_2$. Then $m = \Gamma_1^+ + \Gamma_2^+ + \sigma_1^- + \sigma_2^-$ and $n = \Gamma_1^- + \Gamma_2^- + \sigma_1^+ + \sigma_2^+$, so we define H_i^M as follows:
  - if $i \leq \Gamma_1^-$, then $H_i^M = H_i^{M_1}$;
  - if $i = \Gamma_1^- + j$, with $j \leq \Gamma_2^-$, then $H_i^M = H_j^{M_2}$;
  - if $i = \Gamma^- + j$, with $j \leq \sigma_1^+$, then $H_i^M = H_{\Gamma_1^- + j}^{M_1}$;
  - if $i = \Gamma^- + \sigma_1^+ + j$, with $j \leq \sigma_2^+$, then $H_i^M = H_{\Gamma_2^- + j}^{M_2}$.
- if $M = \mathbf{let}\ \star\ \mathbf{be}\ M\ \mathbf{in}\ N$, then the definition goes as for $(\lambda x.N)M$;
- if $M = \mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ N$, then the definition goes as for $(\lambda x.N)M$.

That $[\![M]\!]^{\mathbf{Int(MetCppo)}} = \bigotimes_j [\![H_j^M]\!]^{\mathbf{MetCppo}}$ can easily be checked by induction on M. ◀

Intuitively, the variables occurring in the left-hand of $\Gamma_j \vdash H_j^M : \mathbf{U}$ correspond to the left-hand "wires" of the string diagram representation of $[\![M]\!]^{\mathbf{Int(MetCppo)}}$, and the first-order term H_j^M describes what exits from i-th right-hand "wire" of $[\![M]\!]^{\mathbf{Int(MetCppo)}}$.

▶ **Example 23.** Let $M = \overline{f}(x(y\overline{0}), z\overline{2})$ and $N = \overline{g}(x(z\overline{1}), y\overline{3})$, so that $\Gamma \vdash M, N : \mathbf{R}$, where $\Gamma = \{x : \mathbf{R} \multimap \mathbf{R}, y : \mathbf{R} \multimap \mathbf{R}, z : \mathbf{R} \multimap \mathbf{R}\}$. The string diagram representations of M and N, with the associated int-terms, are illustrated in Fig. 9.

From Proposition 22 we can now deduce the following:

▶ **Corollary 24.** *For all β-normal terms M, N, $d^{\mathrm{int}}(M, N) = \sum_{j=1}^n d^{\mathrm{den}}(H_j^M, H_j^N)$.*

For instance, in the case of Example 23, the distance $d^{\mathrm{int}}(M, N)$ coincides with the sum of the distances, computed in **MetCppo**, between the int-terms illustrated in Fig. 9.

We can use Corollary 24 to show that the equality $d^{\mathrm{int}} = d^{\mathrm{den}}$ cannot hold. For instance, while $d^{\mathrm{den}}_{(k:\mathbf{R} \multimap \mathbf{I}),\mathbf{I}}(k\overline{2}, k\overline{3}) = 0$, by computing the int-terms $H_1^{k\overline{2}}(x) = H_1^{k\overline{3}}(x) = x$, $H_2^{k\overline{3}} = \overline{2}$, $H_2^{k\overline{3}} = \overline{3}$ we deduce $d^{\mathrm{int}}_{(k:\mathbf{R} \multimap \mathbf{I}),\mathbf{I}}(k\overline{2}, k\overline{3}) = 0 + 1 = 1$.

It remains to prove then that $d^{\mathrm{den}} \leq d^{\mathrm{int}}$.

▶ **Theorem 25.** *For all M, N such that $\Gamma \vdash M, N : \sigma$ holds, $d^{\mathrm{den}}(M, N) \leq d^{\mathrm{int}}(M, N)$.*

▶ **Example 26.** For the terms M and N from Example 23, the procedure just sketched defines the sequence: $M = \overline{f}(x(y\overline{0}), z\overline{2}) \stackrel{\overline{f}(x,z) \mapsto \overline{g}(x,y)}{\to} \overline{g}(x(y\overline{0}), y\overline{0}) \stackrel{y \mapsto z}{\to} \overline{g}(x(z\overline{2}), y\overline{0}) \stackrel{\overline{0} \mapsto \overline{3}}{\to} \overline{g}(x(z\overline{2}), y\overline{3}) \stackrel{\overline{2} \mapsto \overline{1}}{\to} \overline{g}(x(z\overline{1}), y\overline{3}) = N$, where at each step the replacement is of the form $H_i^M[\dots \varphi_j M \dots] \mapsto H_i^N[\dots \varphi_j M \dots]$.

While the argument above holds in the linear case, it does not seem to scale to graded exponentials, and in this last case we are not even sure if a result like Theorem 25 may actually hold (see also the discussion in the next section).

The rest of this section is devoted to prove Theorem 25. For simplicity, we will restrict ourselves to a linear language without unit and tensor types $\mathbf{I}, \sigma \otimes \tau$, and without the associated term constructors. However, the argument developed below can be easily adapted to include such constructors. Given our restriction, we can suppose w.l.o.g. that in Theorem 25 the right-hand type σ is $\mathbf{R}$.

Moreover, it suffices to prove the claim for M, N β-normal, using the fact that, if M^* and N^* are the β-normal forms of M, N, then $d^{\mathrm{den}}(M, M^*) = d^{\mathrm{den}}(N, N^*) = 0$, and moreover $d^{\mathrm{int}}(M^*, N^*) \leq d^{\mathrm{int}}(M, N)$, as a consequence of the non-expansiveness of the trace operator.

Recall that

$$d^{\mathrm{den}}(M, N) = \sup\{d^{\mathrm{den}}_{\sigma}([\![M]\!]^{\mathbf{MetCppo}}(\vec{a}), [\![N]\!]^{\mathbf{MetCppo}}(\vec{a})) \mid \vec{a} \in [\![\Gamma]\!]^{\mathbf{MetCppo}}\}$$

$$d^{\mathrm{int}}(M, N) = \sup\left\{\sum_{i=1}^{n} d^{\mathrm{den}}_{\mathbf{R}}(H_i^M[\vec{r}], H_i^N[\vec{r}]) \;\middle|\; \vec{r} \in \mathbb{R}^m\right\}$$

For fixed $\vec{a} \in [\![\Gamma]\!]^{\mathbf{MetCppo}}$ we will construct reals $\vec{r} \in \mathbb{R}^m$, a sequence of terms $M = M_0, \dots, M_k = N$, where $k = \Gamma^- + \sigma^+$, and a bijection $\rho : \{1, \dots, k\} \to \{1, \dots k\}$ such that the distance between $M_i[\vec{a}]$ and $M_{i+1}[\vec{a}]$ is bounded by the distance between the int-terms $H^M_{\rho(i+1)}[\vec{r}]$ and $H^N_{\rho(i+1)}[\vec{r}]$. In this way we can conclude by a finite number of applications of the triangular law that

$$\begin{aligned} d^{\mathrm{den}}_{\sigma}(M[\vec{a}], N[\vec{a}]) &\leq d^{\mathrm{den}}_{\sigma}(M_0[\vec{a}], M_1[\vec{a}]) + \cdots + d^{\mathrm{den}}_{\sigma}(M_{k-1}[\vec{a}], M_k[\vec{a}]) \\ &\leq d^{\mathrm{den}}_{\mathbf{R}}(H^M_{\rho(1)}[\vec{r}], H^N_{\rho(1)}[\vec{r}]) + \cdots + d^{\mathrm{den}}_{\mathbf{R}}(H^M_{\rho(k)}[\vec{r}], H^N_{\rho(k)}[\vec{r}]) \leq d^{\mathrm{int}}(M, N) \end{aligned}$$

To construct the sequence $M_0, \dots, M_k$, we need a few preliminary results.

For any type σ (or list of types Γ), we indicate as $\{\sigma^+\}$ (resp. $\{\sigma\}^-$) the list, read from left to right, of positive (resp. negative) atomic occurrences in σ. Observe that σ^+ (resp. σ^-) coincides with the length of the list $\{\sigma^+\}$ (resp. $\{\sigma^-\}$).

We will establish a few bijections, more precisely:

- between the elements of the list $\{\Gamma^-\} * \mathbf{R}$ and the *positive subterms* of M (resp. of N), cf. Def. 27 below; this will allow us to associate each first-order term H_i^M with a positive subterm of M;
- between the elements of the list $\{\Gamma^+\}$ and the free and bound variables of M (resp. of N); this will allow us to associate each variable x_i in M with a first-order variable x_i appearing in the int-terms of M.
- finally, between $\{\Gamma^-\} * \mathbf{R}$ and a certain quotient over the set of variables of M.

▶ **Notation 8.1.** *In the following we use $F(x_1, \dots, x_n)$ and $G(x_1, \dots, x_n)$ to indicate linear first-order terms with free variables included in $x_1, \dots, x_n$. Moreover, given terms $M_1, \dots, M_n$ of type R, we indicate with $F(M_1, \dots, M_n)$ the (non necessarily first-order) term obtained by substituting $x_1, \dots, x_n$ with $M_1, \dots, M_n$ in F.*

▶ **Definition 27.** *A subterm of M of the form $N = F(N_1, \dots, N_k)$ is called a* positive subterm of M *if for no other first-order function $\overline{g}$, N occurs in M in a term of the form $\overline{g}(P_1, \dots, P_{r-1}, N, P_{r+1}, \dots, P_k)$. We let $\mathsf{PS}(M)$ indicate the set of positive subterms of M. Moreover, we let $\mathsf{PS}_0(M) \subseteq \mathsf{PS}(M)$ indicate the set of positive subterms of M containing no free or bound variable.*

▶ **Notation 8.2.** *In the following, when indicating positive subterms as $F(M_1,\dots,M_n)$ we make w.l.o.g. the assumption that the terms $M_1,\dots,M_n$ do not start with a function symbol, i.e. are of the form $xQ_1\dots Q_s$.*

▶ **Lemma 28.** *There exists a bijection $\iota_M : \{\Gamma^-\} * \mathbf{R} \to \mathsf{PS}(M)$ (and similarly for N).*

Proof. By induction on M:

- if $M = F(x_1,\dots,x_n)$ is a first-order term, then $\{\Gamma^+\} = \mathbf{R} * \dots * \mathbf{R}$ so $\{\Gamma^-\} * \mathbf{R} = \{\mathbf{R}\}$, and the bijection is $\iota_M(1) = t$;
- if $M = F(M_1,\dots,M_n)$, where $M_i = x_i M_{i1}\dots M_{iq_i}$, then let $M_{ij} = \lambda z_1.\dots.\lambda z_{n_{ij}}.t'_{ij}$, where for some context $\Delta_{ij} = \{z_1 : \sigma_{ij1},\dots,z_{n_{ij}} : \sigma_{ijr_{ij}}\}$, $\Gamma_{ij}, \Delta_{ij} \vdash t'_{ij} : R$, with Γ_{ij} being a partition of $\Gamma - \{x_1 : \sigma_1,\dots,x_n : \sigma_n\}$, with $\sigma_i = \sigma_{i1} \multimap \dots \multimap \sigma_{iq_i} \multimap \mathbf{R}$, with $\sigma_{ij} = \sigma_{ij1} \multimap \dots \multimap \sigma_{ijn_{ij}} \multimap R$; then by the I.H. there exist bijections $\iota_{M_{ij}}$ between $\{(\Gamma_{ij},\Delta_{ij})^-\}$ and $\mathsf{PS}(M_{ij})$. Notice that $\mathsf{PS}(M) = \{t\} \cup \bigcup_{i,j} \mathsf{PS}(M_{ij})$.
  Now, observe that an element of $\{\Gamma^-\} * \mathbf{R}$ is either (1) the last element $\mathbf{R}$, (2) an element of $\{\Gamma^-_{ij}\}$, (3) the last element of some $\{\sigma^+_{ij}\}$, or (4) an element of some $\{\sigma^-_{ijm}\}$. We thus obtain then a bijection $\iota_M : \{\Gamma^-\} * \mathbf{R} \to \mathrm{PS}(M)$ by letting:
  - if l is the last element of $\{\Gamma^-\} * \mathbf{R}$, then $\iota_M(l) = M$;
  - if l is in $\{\Gamma^-_{ij}\}$, $\iota_M(l) = \iota_{M_{ij}}(l)$;
  - if l is the last element of $\{\sigma^+_{ij}\}$, then $\iota_M(l) = M'_{ij}$;
  - if l is in $\{\sigma^-_{ijm}\}$, $\iota_M(l) = \iota_{M_{ij}}(l^*)$, where l^* is the corresponding element in σ_i.

◀

▶ Remark 29. The lemma above actually defines a bijection between the positive subterms of $t = F(N_1,\dots,N_k)$ and the terms H^M_i (which, as described in more detail below, are indeed of the form $F(x_1,\dots,x_k)$).

Let σ be a type; for any occurrence $l \in \{\sigma^+\}$, let σ_l indicate the unique sub-type of σ having l as its rightmost occurrence. Intuitively, $\iota_M(l)$ is the positive subterm that receives type σ_l in the typing of M.

Let $\Gamma = \{x_1 : \sigma_1,\dots,x_n : \sigma_n\}$ and let $\mathrm{V}(M)$ be the set of free and bound variables of t.

▶ **Lemma 30.** *There exists a bijection $\delta_M : \{\Gamma^+\} \to \mathrm{V}(M)$ (and similarly for $\mathrm{V}(N)$).*

Proof. By induction on M:

- if $M = F(x_1,\dots,x_n)$, then $\Gamma^+ = \underbrace{\mathbf{R} * \dots * \mathbf{R}}_{n \text{ times}}$, and we let $\delta_M(i) = x_i$;
- if $M = F(M_1,\dots,M_n)$, where $M_i = x_i M_{i1}\dots M_{iq_i}$, then let $M_{ij} = \lambda z_1.\dots.\lambda z_{n_{ij}}.t'_{ij}$, where for some context $\Delta_{ij} = \{z_1 : \sigma_{ij1},\dots,z_{n_{ij}} : \sigma_{ijr_{ij}}\}$, $\Gamma_{ij}, \Delta_{ij} \vdash t'_{ij} : \mathbf{R}$, with Γ_{ij} being a partition of $\Gamma - \{x_1 : \sigma_1,\dots,x_n : \sigma_n\}$, with $\sigma_i = \sigma_{i1} \multimap \dots \multimap \sigma_{iq_i} \multimap \mathbf{R}$, with $\sigma_{ij} = \sigma_{ij1} \multimap \dots \multimap \sigma_{ijn_{ij}} \multimap \mathbf{R}$; then by the I.H. there exist bijections $\delta_{M_{ij}}$ between $(\Gamma_{ij} * \Delta_{ij})^+$ and $\mathrm{V}(M_{ij})$. Notice that $\mathrm{V}(M) = \{x_1,\dots,x_n\} \cup \bigcup_{i,j} \mathrm{V}(M_{ij})$.
  Now, observe that an element of Γ^+ is either (1) an element of Γ^+_{ij}, (2) the last element of some σ^+_i, or (3) an element of some σ^+_{ijm}. We obtain then a bijection $\iota_M : \Gamma^+ \to \mathrm{V}(M)$ by letting:
  - if l is in Γ^+_{ij}, $\delta_M(l) = \delta_{M_{ij}}(l)$;
  - if l is the last element of σ^+_i, then $\delta_M(l) = x_i$;
  - if l is in σ^+_{ijm}, $\delta_M(l) = \delta_{M_{ij}}(l^*)$, where l^* is the corresponding element in σ_i.

◀

▶ Remark 31. The lemma above actually defines a bijection between the variables of M and the first-order variables appearing in the int-terms of M (which are indeed enumerated by the list $\{\Gamma^+\}$).

▶ **Notation 8.3.** *Using the lemma above* $\mathrm{V}(M) \simeq \mathrm{V}(N) \simeq \Gamma^+$*. We use* X *to indicate any of these equivalent sets. As modulo renaming we can suppose* $\delta_M = \delta_N$*, from now on, for all* $i \in X$*, we indicate as* x_i *the variable* $\delta_M(i) = \delta_N(i)$*.*

▶ **Definition 32.** *For any* $i \in X$*, let* $\phi_i M$ *be the unique subterm of* M *of the form* $x_i M_1 \dots M_k$*, and* $\psi_i M$ *the unique positive subterm of* M *of the form* $F(N_1, \dots, N_{m-1}, \phi_i M, N_{m+1}, \dots, N_n)$*. For all* $i, j \in X$*, let* $i \sqsubset_M j$ *if* $\phi_i M$ *is a subterm of* $\phi_j M$*, and* $i \sim_M j$ *if* $\psi_i M = \psi_j M$*.*

▶ **Lemma 33.** $\sqsubseteq_M$ *is a well-founded order on* X*.*
$\sim_M$ *is an equivalence relation on* X*.*

The relation $\sqsubseteq_M$ extends naturally to $\mathsf{PS}_0(M)$, by letting $F \sqsubset_M i$, for $F \in \mathsf{PS}_0(M)$, if F is a subterm of $\phi_i M$.

Let $X_{\sim_M}$ indicate the quotient of X under $\sim_M$ and let $X^M := X_{\sim_M} \cup \mathsf{PS}_0(M)$. In the following we will use $\xi, \chi, \dots$ to indicate elements of X^M.

Let us extend the relation $\sqsubseteq_M$ from X to X^M:

▶ **Definition 34.** *For all* $\xi, \chi \in X^M$*,* $\xi \sqsubseteq^*_M \chi$ *holds if either* $\xi = \chi$ *or* $\exists j \in \chi\ \forall i \in \xi\ i \sqsubset_M j$*.*

Moreover, we write $\xi \sqsubseteq^0_M \chi$ *if* $\xi \neq \chi$*,* $\xi \sqsubseteq_M \chi$ *and for all* $\theta \in X^M$*,* $\xi \sqsubseteq_M \theta$ *and* $\theta \sqsubseteq_M \chi$ *implies* $\theta \in \{\xi, \chi\}$*.*

Observe that $\xi \sqsubseteq^0_M \chi$ holds precisely when there is $j \in \chi$ such that for all $i \in \xi$, $\phi_j M = x_j Q_1 \dots Q_{l-1}(\psi_i M) Q_{l+1} \dots Q_r$. Moreover the following is easily proved:

▶ **Lemma 35.** $\sqsubseteq^*_M$ *is a well-founded preorder with a maximum element* $\top_M = \{i_1, \dots, i_r\}$*, where* $M = F(\phi_{i_1} M, \dots \phi_{i_r} M)$*.*

We can define a bijection $\theta_M : \mathsf{PS}(M) \to X^M$ sending each positive subterm $P = F(N_1, \dots, N_n)$, where $N_i = x_i Q_{i1} \dots Q_{in_i}$, onto the equivalence class $\{x_1, \dots, x_n\}$, if $n > 0$, and onto the singleton $\{P\}$ otherwise (i.e. if $P \in \mathsf{PS}_0$). The existence of bijections $\theta_M : \mathsf{PS}(M) \to X^M$, $\theta_N : \mathsf{PS}(N) \to X^N$ together with $\iota_M : \{\Gamma^-\} * \mathbf{R}$ and $\iota_M : \{\Gamma^-\} * \mathbf{R}$ implies that the two sets $X^M = X_{\sim_M} \cup \mathsf{PS}_0(M)$ and $X^N = X_{\sim_N} \cup \mathsf{PS}_0(N)$ are also in bijection (being both in bijection with $\{\Gamma^-\} * \mathbf{R}$).

Let $L = (\theta_N \circ \delta_N) \circ (\theta_M \circ \delta_M)^{-1} : X^M \to X^N$ be the bijection associating each element of X^M with the unique element of X^N corresponding to the same occurrence of R in $\{\Gamma^-\} * \mathbf{R}$. Observe that the rightmost element of the list $\{\Gamma^-\} * \mathbf{R}$ is associated with $\top_M = M$ and $\top_N = N$, we deduce that $L(\top_M) = \top_N$.

As a consequence of the bijections established above, we can enumerate the int-terms of M and N using, as index set, X^M rather than $\{\Gamma^-\} * \mathbf{R}$. In particular, for all $\xi \in X^M$, we indicate the associated positive subterm of t as

$$M_\xi := F_\xi \big(\phi_j M\big)_{j \in \xi}$$
$$N_\xi := G_\xi \big(\phi_j N\big)_{j \in L(\xi)}$$

and the associated int-terms as

$$H^M_\xi := F_\xi (x_j)_{j \in \xi}$$
$$H^N_\xi := G_\xi (x_j)_{j \in L(\xi)}$$

We now introduce a special class of terms:

▶ **Definition 36** (terms with brackets)**.** *The set of* λ-terms with brackets *is defined by enriching the syntax of* λ*-terms with a new clause: if* t *is a term, then* $[t]$ *is a term. For any term with brackets* M*, we let* $M^{\downarrow}$ *indicate the term obtained by erasing all brackets from* M*.*

For all $i \in X$, let $\{\phi_i M\} = x_i(\lambda \vec{z}_1.[M_1]) \dots (\lambda \vec{z}_n.[M_n])$, where $\phi_i M = x_i(\lambda \vec{z}.M_1) \dots (\lambda \vec{z}.M_n)$.

▶ **Definition 37.** *A set $U \subseteq X^M$ is* upward closed *if $\alpha \in U$ and $\alpha \sqsubseteq_M \alpha'$ implies $\alpha' \in U$. For all upward closed sets U, the* frontier of U, *noted ∂U, is the set of $\xi \in X^M - U$ such that, for some $\chi \in U$ $\xi \sqsubseteq^0_M \chi$. We conventionally let $\partial\emptyset = \{\top_M\}$.*

For any upward closed $U \subseteq X^M$, we define a set $U(M)$ of λ-terms with brackets by induction on M as follows:

- if $U = \emptyset$ and $M = \lambda\vec{z}.M'$, then $U(M) = \{\lambda\vec{z}.[Q] \mid Q \text{ is a } \lambda\text{-term}\}$;
- if $U \neq \emptyset$ (which implies $\top_M \in U$) and $M = \lambda\vec{z}.F\big(\phi_i M\big)_{i\in\top_M}$, then

$$U(M) = \Big\{\lambda\vec{z}.F\big(x_i Q^i_1, \dots, Q^i_{r_i}\big)_{i\in\top_M} \mid Q^i_j \in U^i_j(M^i_j)\Big\}$$

where for all $i \in \top_M$, $\phi_i M = x_i P^i_1 \dots P^i_{r_i}$ and $U^i_j = U \cap X^{P^i_j}$ is an upward closed set of $X^{M^i_j}$.

Intuitively, $P \in U(M)$ if it is a term which is defined like M at all positions corresponding to elements of U, and has a term in brackets at all positions of ∂U.

The following facts are easily established by induction on M:

▶ **Lemma 38.** *i. $P \in X^M(M)$ iff $P = M$.*
ii. if $P \in U(M)$ and P is bracket-free, then $U = X^M$ and $P = M$.

We now have all ingredients to define, by induction, a sequence of terms $S_0, \dots, S_k$ and a sequence of upward closed sets $U_0 \subseteq \dots \subseteq U_k \subseteq X^N$, verifying at each step i that:

a. $S_i \in U_i(N)$;
b. for all $\xi \in \partial U_i$, S_i contains the subterm $[M_{L^{-1}(\xi)}]$ at position ξ.

Let $S_0 := [M] = [M_{\alpha_\top}]$, and $U_0 = \emptyset$. Then $S_0 \in U_0(M)$ certainly holds. Moreover, $\partial U_0 = \{\top_N\}$, and S_0 contains at its root the subterm $[M_{L^{-1}(\top_N)}] = [M_{\top_M}] = [M]$.

Now, to define S_{i+1}, choose $\xi \in \partial U_i$, let $\chi = L^{-1}(\xi)$ and

$$S_{i+1} := S_i\Big([F_\chi\big(\phi_i M\big)_{i\in\chi}] \mapsto G_\xi\big(\{\phi_j M\}\big)_{j\in\xi}\Big)$$

and finally let $U_{i+1} = U_i \cup \{\xi\}$. To check that S_{i+1} is well-defined let us observe that:

- by the induction hypothesis S_i contains the subterm $[M_\chi] = [F_\chi\big(\phi_i M\big)_{i\in\chi}]$ at position ξ;
- if one of the newly introduced variables $j \in \xi$ is bound in S_i, it is never introduced outside the scope of its abstraction λx_j. Indeed, by the induction hypothesis, S_i coincides with N at all positions $\theta \in U_i$; hence, since N has at position ξ the subterm $G_\xi(\phi_j N)_{j\in\xi}$, it follows that any of the variables x_j is in the scope of an abstraction λx_j in S_{i+1} iff it is in the scope of the same abstraction in N.

Now, from $S_i \in U_i(N)$, that $S_{i+1} \in U_{i+1}(N)$ follows from the definition of $U_{i+1}(N)$. Moreover, if $\xi' \in \partial U_{i+1}$, then either $\xi' \in \partial U_i$, so by IH we deduce that S_i contains $[M_{L^{-1}(\xi')}]$, and since $\xi' \neq \xi$, also S_{i+1} does; or $\xi' \sqsubset^0_M \xi$, i.e. there is $j \in \xi$ such that $\xi' \sqsubset^0_M j$; now, position ξ' in S_{i+1} contains one of the terms Q_i, for $i = 1, \dots, s$, where $[\phi_j M] = x_j W_1 \dots W_s$ and $W_i = \lambda\vec{z}.Q_i$, and indeed we have that $Q_i = [M_{L^{-1}(\xi')}]$.

Let us define now $M_1, \dots, M_k$ as $M_i := S_i^{\downarrow}$. It is clear then that $M_1 = M$. Let us check that $N_k = N$: from $\sharp U_0 = 0$ and $\sharp U_{i+1} = \sharp U_i + 1$, we deduce $\sharp U_k = k$ and thus $\sharp\partial U_k = 0$, which implies $U_k = X^N$. From $S_k \in U_k(N) = X^N(N)$, using Lemma 38 we deduce then that $M_k = S_k^{\downarrow} = S_k = N$.

We can now establish the main result:

▶ **Proposition 39.** *For all $\vec{a} \in [\![\Gamma]\!]$ there exist $\vec{r} \in [\![\Gamma]\!]_+$ such that*

$$|[\![M]\!](\vec{\varphi}) - [\![N]\!](\vec{a})| \;\leq \sum_{i \in \{\Gamma^-\} * R} [\![H_i^M]\!](\vec{r}) - [\![H_i^N]\!](\vec{r})|.$$

Proof. Using the bijection $\delta : \Gamma^+ \to V(M)$, let $r_l := [\![\phi_{\delta(l)} M]\!](\vec{\varphi})$.

Recall that the terms H_i^M and H_i^N, for $i \in \{\Gamma^-\} * \mathbf{R}$, can be equivalently enumerated as $H^M_{L^{-1}(\xi)}$, H^N_ξ, for $\xi \in X^N$. Let $\xi_0, \ldots, \xi_k$ be the sequence in X^N chosen in the construction of the sequence $S_0, \ldots, S_k$, and let $\chi_0 = L^{-1}(\xi_0), \ldots, \chi_k = L^{-1}(\xi_k)$. We will show that for all $i = 1, \ldots, k$, $|[\![M_{i-1}]\!](\varphi) - [\![M_i]\!](\varphi)| \leq |[\![H^M_{\chi_i}]\!](\vec{r}) - [\![H^N_{\xi_i}]\!](\vec{r})|$. Indeed we have

$$\begin{aligned}
\Big|[\![M_{i-1}]\!](\vec{\varphi}) - [\![M_i]\!](\vec{a})\Big| &= \Big|[\![M_{i-1}]\!](\vec{\varphi}) - [\![M_{i-1}\Big(F_{\chi_i}(r_l)_{l \in \chi_i} \mapsto G_{\xi_i}(r_m)_{m \in \xi_i}\Big)]\!](\vec{a})\Big| \\
&= \Big|[\![M_{i-1}]\!](\vec{\varphi}) - [\![M_{i-1}\Big(H^M_{\chi_i}(\vec{r}) \mapsto H^N_{\xi_i}(\vec{r})\Big)]\!](\vec{a})\Big| \\
&\leq \big|[\![H^M_{\chi_i}]\!](\vec{r}) - [\![H^N_{\xi_i}]\!](\vec{r})\big|
\end{aligned}$$

Using the fact that $M_0 = M$ and $M_k = N$, as well as the triangular law, we deduce then

$$\begin{aligned}
|[\![M]\!](\vec{a}) - [\![N]\!](\vec{a})| &\leq |[\![M_0]\!](\vec{a}) - [\![M_1]\!](\vec{a})| + \cdots + |[\![M_{k-1}]\!](\vec{a}) - [\![M_k]\!](\vec{a})| \\
&\leq \sum_{i=1}^{k} \big|[\![H^M_{\chi_i}]\!](\vec{r}) - [\![H^N_{\xi_i}]\!](\vec{r})\big|.
\end{aligned}$$

◀

9 A Linear Programming Language with Graded Exponentials

In the following part of this paper, we generalize some of our arguments to a restriction of Fuzz, namely, Fuzz without additive (co)products and recursive types where gradings are non-negative possibly infinite integers rather than real numbers. We note that while we do not have recursive types, we have recursion. Our generalization goes as follows.

- We describe our target language, which we call $\Lambda^!_S$.
- We extend the logical metric and the observational metric to $\Lambda^!_S$, and we show that these extensions coincide.
- We extend the denotational metric and the interactive metric to $\Lambda^!_S$, and we show that the observational metric is bounded by these metrics.

9.1 Syntax

Let us give our extended target language, which we call $\Lambda^!_S$, and its operational semantics. For types, we have graded exponentials.

$$\text{Types} \qquad \tau, \sigma := \cdots \mid !_n \tau \qquad\qquad \text{Environments} \qquad \Gamma, \Delta, \Xi := \varnothing \mid \Gamma, x :_n \tau$$

In the definition of types and environments, n varies over the set $\mathbb{N}^\infty_{>0} = \{n \in \mathbb{N} \mid n > 0\} \cup \{\infty\}$ consisting of positive possibly infinite integers.

For an environment Γ, we write $|\Gamma|$ for the syntactic object obtained by removing all gradings from $x :_n \tau$ in Γ. For environments Γ and Δ such that $|\Gamma| = |\Delta|$, we inductively define an environment $\Gamma + \Delta$ by

$$\varnothing + \varnothing = \varnothing, \qquad (\Gamma, x :_n \tau) + (\Delta, x :_m \tau) = (\Gamma + \Delta), x :_{n+m} \tau.$$

$$\frac{a \in \mathbb{R}}{\Gamma \vdash \overline{a} : \mathbf{R}} \qquad \frac{}{\Gamma \vdash * : \mathbf{I}} \qquad \frac{f \in S \quad \Gamma_1 \vdash M_1 : \mathbf{R} \quad \dots \quad \Gamma_{\mathrm{ar}(f)} \vdash M_{\mathrm{ar}(f)} : \mathbf{R}}{\Gamma_1 + \cdots + \Gamma_{\mathrm{ar}(f)} \vdash \overline{f}(M_1, \dots, M_{\mathrm{ar}(f)}) : \mathbf{R}}$$

$$\frac{x :_n \tau \in \Gamma \quad n \geq 1}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma \vdash M : \tau \quad n \cdot \Gamma \leq \Delta}{\Delta \vdash !M : !_n \tau} \qquad \frac{\Gamma, f :_\infty \tau \multimap \sigma, x :_1 \tau \vdash M : \sigma}{\infty \cdot \Gamma \vdash \mathbf{fix}_{\tau,\sigma}(f, x, M) : \tau \multimap \sigma}$$

$$\frac{\Gamma \vdash M : \mathbf{R} \quad \Delta \vdash N : \mathbf{R}}{\Gamma + \Delta \vdash M + N : \mathbf{R}} \qquad \frac{\Gamma \vdash M : !_n \mathbf{R} \quad |a| \leq n}{\Gamma \vdash \overline{a} \cdot M : \mathbf{R}}$$

$$\frac{\Gamma, x :_1 \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma. M : \sigma \multimap \tau} \qquad \frac{\Gamma \vdash M : \sigma \multimap \tau \quad \Delta \vdash N : \sigma}{\Gamma + \Delta \vdash M\, N : \tau} \qquad \frac{\Gamma \vdash M : \tau \quad \Delta \vdash N : \sigma}{\Gamma + \Delta \vdash M \otimes N : \tau \otimes \sigma}$$

$$\frac{\Gamma \vdash M : \mathbf{I} \quad \Xi \vdash N : \tau \quad n \cdot \Gamma \leq \Delta}{\Delta + \Xi \vdash \mathbf{let}\ *\ \mathbf{be}\ M\ \mathbf{in}\ N : \tau} \qquad \frac{\Gamma \vdash M : \sigma_1 \otimes \sigma_2 \quad \Xi, x :_n \sigma_1, y :_n \sigma_2 \vdash N : \tau \quad n \cdot \Gamma \leq \Delta}{\Delta + \Xi \vdash \mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ N : \tau}$$

$$\frac{\Gamma \vdash M : !_m \sigma \quad \Xi, x :_{n \cdot m} \sigma \vdash N : \tau \quad n \cdot \Gamma \leq \Delta}{\Delta + \Xi \vdash \mathbf{let}\ !x\ \mathbf{be}\ M\ \mathbf{in}\ N : \tau}$$

Figure 10 Typing Rules

When we write $\Gamma + \Delta$, we always suppose that $|\Gamma|$ is equal to $|\Delta|$. We write $\Gamma \geq \Delta$ when $|\Gamma| = |\Delta|$ and for all $x :_n \tau \in \Gamma$ and $x :_m \tau \in \Delta$, we have $n \leq m$. For an environment Γ and $n \in \mathbb{N}^\infty_{>0}$, we define $n \cdot \Gamma$ to be the environment obtained by the componentwise multiplication of gradings in Γ by n.

Terms, values contexts are given by the following BNF.

$$\begin{aligned}
\text{Terms} \quad & M, N := \cdots \mid !M \mid \mathbf{let}\ !x\ \mathbf{be}\ M\ \mathbf{in}\ N \mid \mathbf{fix}_{\tau,\sigma}(f, x, M) \mid \overline{a} \cdot M \mid M + N \\
\text{Values} \quad & V, U := \cdots \mid \mathbf{fix}_{\tau,\sigma}(f, x, M) \mid !V \\
\text{Contexts} \quad & C[-] := \cdots \mid !C[-] \mid \mathbf{let}\ !x\ \mathbf{be}\ C[-]\ \mathbf{in}\ M \mid \mathbf{let}\ !x\ \mathbf{be}\ M\ \mathbf{in}\ C[-]
\end{aligned}$$

Namely, we have graded exponentials $!(-)$, let-bindings for the graded comonad, recursion, unary multiplications and addition. We add these term constructors so as to simplify the definition of the observational metric on $\Lambda^!_S$. Typing rules are given in Figure 10, and evaluation rules are given in Figure 11. We naturally extend the definition of $\mathbf{Term}^!(\Gamma, \tau)$, $\mathbf{Value}^!(\tau)$ and $\mathbf{Value}^!(\Gamma)$ to $\Lambda^!_S$. We write $C[-]\colon (\Gamma, \tau) \to (\Delta, \sigma)$ when $C[-]$ satisfies the following conditions.

- For all terms $\Gamma \vdash M : \tau$, we have $\Delta \vdash C[M] : \sigma$.
- For a fresh variable y, we have $y :_1 !_{k_1}\tau_1 \multimap \cdots !_{k_n}\tau_n \multimap \tau, \Delta \vdash C[y\, !x_1 \cdots !x_n] : \sigma$ where $\Gamma = (x_1 :_{k_1} \tau_1, \dots, x_n :_{k_n} \tau_n)$.

We do not have **fix** in the definition of contexts because when a hole of a context is under **fix**, then the second condition never holds. Intuitively, the second condition means that the hole $[-]$ appears linearly in the context $C[]$.

The following propositions can be shown by induction on derivations of type judgements.

▶ **Proposition 40** (Substitution). *If $\Gamma \vdash M : \tau$ and $\gamma \in \mathbf{Value}^!(\Gamma)$, then $\vdash M\gamma : \tau$.*

▶ **Proposition 41** (Preservation). *If $\vdash M : \tau$ and $M \hookrightarrow V$, then $\vdash V : \tau$.*

In general, a type judgement $\Gamma \vdash M : \tau$ may have different derivations. For example,

$$\frac{}{V \hookrightarrow V} \qquad \frac{M_1 \hookrightarrow \overline{a_1} \quad \dots \quad M_n \hookrightarrow \overline{a_n}}{\overline{f}(M_1,\dots,M_n) \hookrightarrow \overline{f(a_1,\dots,a_n)}} \qquad \frac{M \hookrightarrow \overline{a} \quad N \hookrightarrow \overline{b} \quad c = a+b}{M+N \hookrightarrow \overline{c}}$$

$$\frac{M \hookrightarrow !\overline{b} \quad ab = c}{\overline{a}\cdot M \hookrightarrow \overline{c}} \qquad \frac{M \hookrightarrow \lambda x:\tau.\, M' \quad N \hookrightarrow V \quad M'[V/x] \hookrightarrow U}{M\,N \hookrightarrow U}$$

$$\frac{M \hookrightarrow V \quad N \hookrightarrow U}{M \otimes N \hookrightarrow V \otimes U} \qquad \frac{M \hookrightarrow * \quad N \hookrightarrow V}{\mathbf{let}\ *\ \mathbf{be}\ M\ \mathbf{in}\ N \hookrightarrow V} \qquad \frac{M \hookrightarrow V \otimes U \quad N[V/x, U/y] \hookrightarrow W}{\mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ N \hookrightarrow W}$$

$$\frac{M \hookrightarrow \mathbf{fix}_{\tau,\sigma}(f,x,M') \quad N \hookrightarrow V \quad M'[\mathbf{fix}_{\tau,\sigma}(f,x,M')/f, V/x] \hookrightarrow U}{M\,N \hookrightarrow U}$$

$$\frac{M \hookrightarrow V}{!M \hookrightarrow !V} \qquad \frac{M \hookrightarrow !V \quad N[V/x] \hookrightarrow U}{\mathbf{let}\ !x\ \mathbf{be}\ M\ \mathbf{in}\ N \hookrightarrow U}$$

Figure 11 Evaluation Rules

$x :_3 \mathbf{R} \vdash x + x : \mathbf{R}$ has the following derivations.

$$\frac{x :_1 \mathbf{R} \vdash x : \mathbf{R} \quad x :_2 \mathbf{R} \vdash x : \mathbf{R}}{x :_3 \mathbf{R} \vdash x + x : \mathbf{R}}, \qquad \frac{x :_2 \mathbf{R} \vdash x : \mathbf{R} \quad x :_1 \mathbf{R} \vdash x : \mathbf{R}}{x :_3 \mathbf{R} \vdash x + x : \mathbf{R}}.$$

We can show that grading is the only source of non-uniqueness of derivations. This observation is useful to define denotational semantics for $\Lambda^!_S$ in Section 12.

▶ **Proposition 42.** *For any environment Γ and any term M, if D_1 is a derivation of $\Gamma \vdash M : \tau$ and D_2 is a derivation of $\Gamma \vdash M : \sigma$, then τ can be obtained by changing gradings in σ, and D_1 can also be obtained by changing gradings in D_2.*

10 Logical Metric and Observational Metric

10.1 Metric Logical Relation

We define metric logical relations

$$\{(-) \preceq^\tau_r (-) \subseteq \mathbf{Term}^!(\tau) \times \mathbf{Term}^!(\tau)\}_{\tau \in \mathbf{Ty},\ r \in \mathbb{R}^\infty_{\geq 0}}$$

for $\Lambda^!_S$ by induction on τ as follows.

$$\begin{aligned}
M \preceq^{\mathbf{R}}_r N &\iff \text{if } M \hookrightarrow a,\ \text{then } N \hookrightarrow b \text{ and } |a-b| \leq r \\
M \preceq^{\mathbf{I}}_r N &\iff \text{if } M \hookrightarrow *,\ \text{then } N \hookrightarrow * \\
M \preceq^{\tau\otimes\sigma}_r N &\iff \text{if } M \hookrightarrow V \otimes V',\ \text{then } N \hookrightarrow U \otimes U' \\
&\qquad \text{and } \exists s, s' \in \mathbb{R}^\infty_{\geq 0},\ V \preceq^\tau_s U \text{ and } V' \preceq^\sigma_{s'} U' \text{ and } s + s' \leq r \\
M \preceq^{\tau\multimap\sigma}_r N &\iff \text{if } M \hookrightarrow V,\ \text{then } N \hookrightarrow V' \\
&\qquad \text{and } \forall U, U' \in \mathbf{Value}^!(\tau),\ \text{if } U \preceq^\tau_s U',\ \text{then } V\,U \preceq^\sigma_{r+s} V'\,U' \\
M \preceq^{!_n\tau}_r N &\iff \text{if } M \hookrightarrow !V,\ \text{then } N \hookrightarrow !U \\
&\qquad \text{and } \exists s \in \mathbb{R}^\infty_{\geq 0},\ V \precsim^\tau_s U \text{ and } r \geq n\,s.
\end{aligned}$$

Let $\Gamma = (x_1 :_{k_1} \sigma_1, \dots, x_n :_{k_n} \sigma_n)$ be an environment. For $\gamma = (V_1, \dots, V_n)$ and $\gamma' = (V'_1, \dots, V'_n)$ in $\mathbf{Value}^!(\Gamma)$, and for $\epsilon = (r_1, \dots, r_n) \in (\mathbb{R}^\infty_{\geq 0})^n$, we write $\gamma \preceq^\Gamma_\epsilon \gamma'$ when we have

$V_1 \preceq^{\sigma_1}_{r_1} V_1', \ldots, V_n \preceq^{\sigma_n}_{r_n} V_n'$. We define $\epsilon \cdot \Gamma$ to be $r_1 k_1 + \cdots + r_n k_n$. Here, we define 0∞ to be 0. Then, for terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we define $d^{\log}_{\Gamma,\tau}(M,N) \in \mathbb{R}^{\infty}_{\geq 0}$ by

$$d^{\log}_{\Gamma,\tau}(M,N) = \inf \left\{ r \in \mathbb{R}^{\infty}_{\geq 0} \;\middle|\; \begin{array}{l} \forall \gamma, \gamma' \in \mathbf{Value}^!(\Gamma),\text{ if } \gamma \preceq^{\Gamma}_{\epsilon} \gamma',\text{ then} \\ M\gamma \preceq^{\tau}_{r+\epsilon\cdot\Gamma} N\gamma' \text{ and } N\gamma \preceq^{\tau}_{r+\epsilon\cdot\Gamma} M\gamma' \end{array} \right\}.$$

We call $d^{\log}$ the *logical metric* on $\Lambda^!_S$. For later use, we prove the fundamental lemma.

▶ **Lemma 43.** *Let $\Gamma = (x_1 :_{k_1} \tau_1, \ldots, x_n :_{k_n} \tau_n)$ be an environment, and let $\Gamma \vdash M : \tau$ be a term. Given $\gamma, \gamma' \in \mathbf{Value}^!(\Gamma)$ such that $\gamma \preceq^{\Gamma}_{\epsilon} \gamma'$, then we have $M\gamma \preceq^{\tau}_{\epsilon\cdot\Gamma} M\gamma'$.*

Proof. The proof is essentially the same with [25] using step indexed logical relations counting the number of **fix**-reductions in $M \hookrightarrow V$. ◀

11 Observational Metric

For terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we define $d^{\text{obs}}_{\Gamma,\tau}(M,N) \in \mathbb{R}^{\infty}_{\geq 0}$ by

$$d^{\text{obs}}_{\Gamma,\tau}(M,N) = \sup_{C[-] :\, (\Gamma,\tau)\to(\varnothing,\mathbf{R})} \inf \left\{ r \in \mathbb{R}^{\infty}_{\geq 0} \,\middle|\, C[M] \sqsubseteq_r C[N] \text{ and } C[N] \sqsubseteq_r C[M] \right\}$$

where for $\vdash L, L' : \mathbf{R}$,

$$L \sqsubseteq_r L' \iff \text{if } L \hookrightarrow a,\ \text{then } L' \hookrightarrow b \text{ and } |a-b| \leq r.$$

▶ **Theorem 44.** *For terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we have $d^{\text{obs}}_{\Gamma,\tau}(M,N) = d^{\log}_{\Gamma,\tau}(M,N)$.*

Proof. The statement follows from Lemma 45 and Lemma 46 shown below. ◀

▶ **Lemma 45.** *For any environment $\Gamma = (x_1 :_{k_1} \tau_1, \ldots, x_n :_{k_n} \tau_n)$ and any pair of terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, if $\gamma \in \mathbf{Value}^!(\Gamma)$ and $d^{\text{obs}}_{\Gamma,\tau}(M,N) < \infty$, then $M\gamma \preceq^{\tau}_{d^{\text{obs}}_{\Gamma,\tau}(M,N)} N\gamma$.*

Proof. By induction on τ, we show that for all $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we have $M\gamma \preceq^{\tau}_{d^{\text{obs}}_{\Gamma,\tau}(M,N)} N\gamma$. We only give a proof for $\tau = \sigma \otimes \rho$ and $\tau = \sigma \multimap \rho$. (The case of $\tau = \sigma \otimes \rho$) Given $\gamma \in \mathbf{Value}^!(\Gamma)$, if $M\gamma$ diverges, then by the definition of $\preceq$, we obtain $M\gamma \preceq_{d^{\text{obs}}_{\Gamma,\tau}(M,N)} N\gamma$. If we have $M\gamma \hookrightarrow V_1 \otimes V_2$, then since $d^{\text{obs}}_{\Gamma,\tau}(M,N) < \infty$, there are $U_1 \in \mathbf{Value}^!(\sigma)$ and $U_2 \in \mathbf{Value}^!(\rho)$ such that $N\gamma \hookrightarrow U_1 \otimes U_2$. By the induction hypothesis on σ and ρ, we have $V_1 \preceq^{\sigma}_{d^{\text{obs}}_{\varnothing,\sigma}(V_1,U_1)} U_1$ and $V_2 \preceq^{\rho}_{d^{\text{obs}}_{\varnothing,\rho}(V_2,U_2)} U_2$. It remains to check that $d^{\text{obs}}_{\varnothing,\sigma}(V_1,U_1) + d^{\text{obs}}_{\varnothing,\rho}(V_2,U_2) \leq d^{\text{obs}}_{\Gamma,\sigma\otimes\rho}(M,N)$. Below, we suppose that $\gamma = (W_1, \ldots, W_n)$. We write $\widetilde{M}$ and $\widetilde{N}$ for

$$(\lambda x_1 : \tau_1. \cdots \lambda x_n : \tau_n. M)\, W_1 \cdots W_n$$
$$(\lambda x_1 : \tau_1. \cdots \lambda x_n : \tau_n. N)\, W_1 \cdots W_n$$

respectively. Then,

$$
\begin{aligned}
& d^{\mathrm{obs}}_{\varnothing,\sigma}(V_1,U_1) + d^{\mathrm{obs}}_{\varnothing,\rho}(V_2,U_2) \\
\leq & \sup_{\substack{C[-]\colon(\varnothing,\sigma)\to(\varnothing,\mathbf{R}) \\ D[-]\colon(\varnothing,\rho)\to(\varnothing,\mathbf{R})}} \inf \left\{ r\in\mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{l} C[U_1]+D[U_2] \sqsubseteq_r C[V_1]+D[V_2] \\ \text{and} \\ C[V_1]+D[V_2] \sqsubseteq_r C[U_1]+D[U_2] \end{array} \right\} \\
\leq & \sup_{\substack{C[-]\colon(\varnothing,\sigma)\to(\varnothing,\mathbf{R}) \\ D[-]\colon(\varnothing,\rho)\to(\varnothing,\mathbf{R})}} \inf \left\{ r\in\mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{l} \mathbf{let}\ x\otimes y\ \mathbf{be}\ \widetilde{M}\ \mathbf{in}\ C[x]+D[y] \\ \qquad \sqsubseteq_r \mathbf{let}\ x\otimes y\ \mathbf{be}\ \widetilde{N}\ \mathbf{in}\ C[x]+D[y] \\ \text{and} \\ \mathbf{let}\ x\otimes y\ \mathbf{be}\ \widetilde{N}\ \mathbf{in}\ C[x]+D[y] \\ \qquad \sqsubseteq_r \mathbf{let}\ x\otimes y\ \mathbf{be}\ \widetilde{M}\ \mathbf{in}\ C[x]+D[y] \end{array} \right\} \\
\leq & \; d^{\mathrm{obs}}_{\Gamma,\sigma\otimes\rho}(M,N).
\end{aligned}
$$

We note that we use the addition to construct contexts. We need unary multiplications to prove the case where $\tau = !_n\sigma$. (The case of $\tau = \sigma \multimap \rho$) Given $\gamma \in \mathbf{Value}^!(\Gamma)$, if $M\gamma$ diverges, then by the definition of $\preceq$, we have $M\gamma \preceq^{\tau}_{d^{\mathrm{obs}}_{\Gamma,\tau}(M,N)} N\gamma$. Let us assume that we have $M\gamma \hookrightarrow V$, and we show that $M\gamma \preceq^{\tau}_{d^{\mathrm{obs}}_{\Gamma,\tau}(M,N)} N\gamma$. From the assumption, since $d^{\mathrm{obs}}_{\Gamma,\tau}(M,N) < \infty$, we see that we have $N\gamma \hookrightarrow V'$ for some $V' \in \mathbf{Value}^!(\tau)$. In order to prove $M\gamma \preceq^{\tau}_{d^{\mathrm{obs}}_{\Gamma,\tau}(M,N)} N\gamma$, we show that for all $U \preceq^{\sigma}_{r} U'$, we have $V\,U \preceq^{\rho}_{r+d^{\mathrm{obs}}_{\Gamma,\tau}(M,N)} V'\,U'$. By Lemma 43, we obtain $V'\,U \preceq^{\rho}_{r} V'\,U'$. Hence, by the triangle inequality, it remains to check $V\,U \preceq^{\rho}_{d^{\mathrm{obs}}_{\Gamma,\tau}(M,N)} V'\,U$. By the definition of $\preceq$, this is equivalent to $M\gamma\,U \preceq^{\rho}_{d^{\mathrm{obs}}_{\Gamma,\tau}(M,N)} N\gamma\,U$. It follows from the induction hypothesis on ρ that we have

$$M\gamma\,U \preceq^{\rho}_{d^{\mathrm{obs}}_{\Gamma,\rho}(M\,U,N\,U)} N\gamma\,U.$$

Since $d^{\mathrm{obs}}_{\Gamma,\rho}(M\,U,N\,U) \leq d^{\mathrm{obs}}_{\Gamma,\tau}(M,N)$, we obtain the claim. ◀

▶ **Lemma 46.** *For terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we have $d^{\mathrm{obs}}_{\Gamma,\tau}(M,N) \leq d^{\mathrm{log}}_{\Gamma,\tau}(M,N)$.*

Proof. For simplicity, we suppose that $\Gamma = (x :_k \sigma)$. We show that if

$$\lambda y : !_k\sigma.\,\mathbf{let}\ !x\ \mathbf{be}\ y\ \mathbf{in}\ M \preceq^{!_k\sigma\multimap\tau}_{r} \lambda y : !_k\sigma.\,\mathbf{let}\ !x\ \mathbf{be}\ y\ \mathbf{in}\ N$$
$$\lambda y : !_k\sigma.\,\mathbf{let}\ !x\ \mathbf{be}\ y\ \mathbf{in}\ N \preceq^{!_k\sigma\multimap\tau}_{r} \lambda y : !_k\sigma.\,\mathbf{let}\ !x\ \mathbf{be}\ y\ \mathbf{in}\ M$$

for some $r \in \mathbb{R}^{\infty}_{\geq 0}$, then $d^{\mathrm{obs}}_{\Gamma,\tau}(M,N) \leq r$. Given a context $C[-]\colon(\Gamma,\tau)\to(\varnothing,\mathbf{R})$, it follows from adequacy of denotational model (Theorem 47) that

$$
\begin{aligned}
C[M] \hookrightarrow a &\iff (\lambda z : !_k\sigma \multimap \tau.\,C[z\,!x])(\lambda y : !_k\sigma.\,\mathbf{let}\ !x\ \mathbf{be}\ y\ \mathbf{in}\ M) \hookrightarrow a, \\
C[N] \hookrightarrow b &\iff (\lambda z : !_k\sigma \multimap \tau.\,C[z\,!x])(\lambda y : !_k\sigma.\,\mathbf{let}\ !x\ \mathbf{be}\ y\ \mathbf{in}\ N) \hookrightarrow b.
\end{aligned}
$$

Hence, it follows from Lemma 43 that $C[M]$ converges if and only if $C[N]$ converges. If $C[M] \hookrightarrow a$ and $C[N] \hookrightarrow b$, then we have $|a-b| \leq r$. Since this holds for any context $C[-]\colon(\Gamma,\tau)\to(\varnothing,\mathbf{R})$, we obtain $d^{\mathrm{obs}}_{\Gamma,\tau}(M,N) \leq r$. ◀

12 Denotational Metric

Let $\mathbf{MetCpo}_{\perp}$ be the category of pointed metric cpos and strict continuous and non-expansive functions. Concretely, objects in $\mathbf{MetCpo}_{\perp}$ are metric cpos with least elements

$\bot$ such that the distances $d(\bot, x)$ are ∞ when $x \neq \bot$, and morphisms from X to Y are bottom-preserving. As is shown in [3], $\mathbf{MetCpo}_\bot$ provides an adequate semantics for Fuzz. By restricting their result to our language, we obtain adequacy for $\Lambda^!_S$. For a term $\Gamma \vdash M : \tau$, let us write $[\![M]\!]^{\mathrm{den}} \colon [\![\Gamma]\!]^{\mathrm{den}} \to [\![\tau]\!]^{\mathrm{den}}$ for the interpretation of M in $\mathbf{MetCpo}_\bot$. We note that $[\![M]\!]^{\mathrm{den}}$ is defined with respect to the type judgement rather than type derivations of $\Gamma \vdash M : \tau$. This can be checked by observing that the underlying continuous function of $[\![M]\!]^{\mathrm{den}}$ is obtained by first transforming $\Lambda^!_S$ into the λ_c-calculus [23] and then interpreting the transformed term in $\mathbf{Cpo}_\bot$. In the transformation of $\Lambda^!_S$ into the λ_c-calculus, gradings are dropped, and therefore, all derivations of a type judgement $\Gamma \vdash M : \tau$ are transformed into the same derivation in the λ_c-calculus.

▶ **Theorem 47** ([3]). *Let $\vdash M : \tau$ be a term in $\Lambda^!_S$.*

- *If $M \hookrightarrow V$, then $[\![M]\!]^{\mathrm{den}} = [\![V]\!]^{\mathrm{den}}$.*
- *If $[\![M]\!]^{\mathrm{den}} \neq \bot$, then there is a value $V \in \mathbf{Value}^!(\tau)$ such that $M \hookrightarrow V$.*

For terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we define $d^{\mathrm{den}}_{\Gamma,\tau}(M, N) \in \mathbb{R}^{\infty}_{\geq 0}$ by

$$d^{\mathrm{den}}_{\Gamma,\tau}(M, N) = d([\![M]\!]^{\mathrm{den}}, [\![N]\!]^{\mathrm{den}}).$$

It is easy to see that d^{den} is a metric on $\Lambda^!_S$. We call d^{den} the *denotational metric* on $\Lambda^!_S$.

It follows from adequacy of $\mathbf{MetCpo}_\bot$ that d^{obs} is bounded by d^{den}.

▶ **Theorem 48.** *$d^{\mathrm{obs}} \leq d^{\mathrm{den}}$.*

Proof. If there is a context $C[-] \colon (\Gamma, \tau) \to (\varnothing, \mathbf{R})$ such that $C[M]$ converges and $C[N]$ diverges, then, by adequacy, we have $[\![C[M]]\!] = [\![\overline{a}]\!]$. $[\![C[N]]\!] = \bot$ for some $a \in \mathbb{R}$. Hence, $d^{\mathrm{den}}_{\Gamma,\tau}(M, N) \geq d^{\mathrm{den}}_{\varnothing,\mathbf{R}}(C[M], C[N]) = \infty \geq d^{\mathrm{obs}}_{\Gamma,\tau}(M, N)$. Similarly, when $C[M]$ diverges and $C[N]$ converges, then $d^{\mathrm{den}}_{\Gamma,\tau}(M, N) \geq d^{\mathrm{obs}}_{\Gamma,\tau}(M, N)$. Below, we suppose that for any context $C[-] \colon (\Gamma, \tau) \to (\varnothing, \mathbf{R})$, $C[M]$ diverges if and only if $C[N]$ diverges. In this case, it follows from Theorem 47 that if $C[M] \hookrightarrow a$ and $C[N] \hookrightarrow b$, then $|a - b| \leq d^{\mathrm{den}}_{\Gamma,\tau}(M, N)$. Hence, we obtain the statement. ◀

13 Interactive Semantic Model

13.1 Preparation

13.1.1 Structures for Interpreting Graded Exponentials

We prepare structures on the category $\mathbf{Int}(\mathbf{MetCppo})$ that we use to interpret graded exponentials in $\Lambda^!_S$. For $X \in \mathbf{MetCppo}$ and $n \in \mathbb{N}^{\infty}_{>0}$, we define $n \cdot X$ to be the countably infinite product of the underlying cpo of X equipped with the following metric:

$$d((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) = \sum_{i < n} d(x_i, y_i).$$

It is not difficult to check that $n \cdot (-)$ is a traced symmetric monoidal functor on $\mathbf{MetCppo}$. Hence, we can lift the functors $n \cdot (-)$ to symmetric monoidal functors on $\mathbf{Int}(\mathbf{MetCppo})$. Abusing notation, we also denote the functors on $\mathbf{Int}(\mathbf{MetCppo})$ by $n \cdot (-)$. To be concrete, on objects $X = (X_+, X_-)$ in $\mathbf{Int}(\mathbf{MetCppo})$, we have $n \cdot X = (n \cdot X_+, n \cdot X_-)$.

In order to interpret dereliction, digging and contraction of $\Lambda^!_S$, for $n, m \in \mathbb{N}^{\infty}_{>0}$, we choose bijections $u_{n,m} \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $v_{n,m} \colon \{0, 1\} \times \mathbb{N} \to \mathbb{N}$ such that

- $u_{n,m}$ embeds $\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i < n \text{ and } j < m\}$ into $\{i \in \mathbb{N} \mid i < nm\}$; and

- $v_{n,m}$ embeds $\{(0,i) \mid i < n\} \cup \{(1,i) \mid i < m\}$ into $\{i \in \mathbb{N} \mid i < n+m\}$.

Then, we define the following morphisms

$$\begin{aligned} d_{n,X} &\colon n \cdot X \to X, \\ \delta_{n,m,X} &\colon nm \cdot X \cong n \cdot (m \cdot X), \\ c_{n,m,X} &\colon (n+m) \cdot X \cong (n \cdot X) \otimes (m \cdot X) \end{aligned}$$

for $n, m \in \mathbb{N}^{\infty}_{>0}$ by

$$\begin{aligned} d_{n,X}((x_i)_{i\in\mathbb{N}}, y) &= ((y, \bot, \bot, \ldots), x_0), \\ \delta_{n,m,X}((x_i)_{i\in\mathbb{N}}, ((y_{i,j})_{j\in\mathbb{N}})_{i\in\mathbb{N}}) &= ((y_{u^{-1}_{n,m}(i)})_{i\in\mathbb{N}}, ((x_{u_{n,m}(i,j)})_{j\in\mathbb{N}})_{i\in\mathbb{N}}), \\ c_X((x_i)_{i\in\mathbb{N}}, ((y_{0,i})_{i\in\mathbb{N}}, (y_{1,i})_{i\in\mathbb{N}})) &= ((y_{v_{n,m}(i)})_{i\in\mathbb{N}}, ((x_{v^{-1}_{n,m}(0,i)})_{i\in\mathbb{N}}, (x_{v^{-1}_{n,m}(1,i)})_{i\in\mathbb{N}})). \end{aligned}$$

We also give a bit more general dereliction $\tilde{d}_{n,m,X} \colon (n+m) \cdot X \to n \cdot X$ by

$$\begin{aligned} \tilde{d}_{n,m,X}((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = ((y_0, y_1, \ldots, y_{n-1}, \overbrace{\bot, \ldots, \bot}^{m}, y_n, y_{n+1}, \ldots), \\ (x_0, x_1, \ldots, x_{n-1}, x_{n+m-1}, x_{n+m}, \ldots)). \end{aligned}$$

We note that morphisms $d_{n,X}$, $\delta_{n,m,X}$, $c_{n,m,X}$, $\tilde{d}_{n,m,X}$ and w_X are not natural with respect to X. Still, we can show that they are *pointwise* natural transformation, that is, these morphisms satisfy naturality conditions for global elements. For example, for all $x \colon I \to X$, we have $d_{n,X} \circ (n \cdot x) = x \circ d_{n,I} = x$. For more details on pointwise naturality, see [1].

13.1.2 Structures for Interpreting Weakening and Cbv Evaluation

We also need structures on **Int**(**MetCppo**) to interpret weakening and call-by-value evaluation. For the former, for an object $X \in$ **Int**(**MetCppo**), we define $w_X \colon X \to I$ in **Int**(**MetCppo**) to be $\bot \colon X_+ \to X_-$ in **MetCppo**. For the latter, we use the Kleisli category of a continuation monad

$$TX = X \otimes (K, K) \cong (X \multimap (K, I)) \multimap (K, I)$$

on **Int**(**MetCppo**) where $K \in$ **MetCppo** is the Sierpiński space $\{\bot \leq \top\}$ equipped with $d(\bot, \top) = \infty$. We denote the unit, the multiplication and the strength of the monad T by

$$\begin{aligned} \eta_X &\colon X \to TX, \\ \mu_X &\colon TTX \to TX, \\ \mathrm{str}_{X,Y} &\colon TX \otimes Y \to T(X \otimes Y), \end{aligned}$$

and we write $\mathrm{dstr}_{X,Y} \colon TX \otimes TY \to T(X \otimes Y)$ for the double strength

$$TX \otimes TY \longrightarrow T(X \otimes TY) \longrightarrow TT(X \otimes Y) \longrightarrow T(X \otimes Y).$$

For $n \in \mathbb{N}^{\infty}_{>0}$, we define a morphism

$$\xi_{n,X} \colon n \cdot TX \to T(n \cdot X)$$

in **Int**(**MetCppo**) to be

$$n \cdot TX = n \cdot (X \otimes (K, K)) \cong n \cdot X \otimes n \cdot (K, K) \xrightarrow{(n \cdot X) \otimes d} (n \cdot X) \otimes (K, K) = T(n \cdot X).$$

We use this distributivity of $n \cdot (-)$ over T to model the action of graded exponentials on terms.

13.1.3 Structures for Interpreting Constants

For interpretation of constants $\overline{a}$ and $\overline{f}(M_1, \ldots, M_{\mathrm{ar}(f)})$, we follow the interpretation of terms in Λ_S: we interpret the base type $\mathbf{R}$ by (R, I), and we use $\lfloor a \rfloor : I \to R$ and $\lfloor f \rfloor : R^{\otimes \mathrm{ar}(f)} \to R$ to interpret real numbers and first order functions. For interpretation of unary multiplications, we use

$$\mathrm{mult}_{a,n} : n \cdot (R, I) \to (R, I)$$

for $a \in \mathbb{R}$ and $n \in \mathbb{N}^{\infty}_{>0}$ such that $|a| \leq n$ given by

$$\mathrm{mult}_{a,n}((x_i)_{i \in \mathbb{N}}, *) = (*, a(x_0 + \cdots + x_{n-1})/n).$$

13.2 Interactive Semantic Model and its Associated Metrics

Based on preparations in the previous sections, we give interpretation of $\Lambda^!_S$ in the Kleisli category $\mathbf{Int}(\mathbf{MetCppo})_T$.

Types in $\Lambda^!_S$ are interpreted as follows:

$$\begin{aligned}
[\![\mathbf{R}]\!]^{\mathrm{int}} &= (R, I), \\
[\![\mathbf{I}]\!]^{\mathrm{int}} &= (I, I), \\
[\![\tau \otimes \sigma]\!]^{\mathrm{int}} &= [\![\tau]\!]^{\mathrm{int}} \otimes [\![\sigma]\!]^{\mathrm{int}} \\
[\![\tau \multimap \sigma]\!]^{\mathrm{int}} &= [\![\tau]\!]^{\mathrm{int}} \multimap T[\![\sigma]\!]^{\mathrm{int}} = ([\![\tau]\!]^{\mathrm{int}})^* \otimes [\![\sigma]\!]^{\mathrm{int}} \otimes (K, K) \\
[\![!_n \tau]\!]^{\mathrm{int}} &= n \cdot [\![\tau]\!]^{\mathrm{int}}
\end{aligned}$$

where $(X_+, X_-)^*$ is defined to be (X_-, X_+). As usual, we interpret environments as follows:

$$[\![(x :_n \tau, \ldots, y :_m \sigma)]\!]^{\mathrm{int}} = n \cdot [\![\tau]\!]^{\mathrm{int}} \otimes \cdots \otimes m \cdot [\![\sigma]\!]^{\mathrm{int}}.$$

We next define interpretation of type judgements in $\Lambda^!_S$ in Figure 12 where we simply write $[\![-]\!]$ for $[\![-]\!]^{\mathrm{int}}$ for the sake of legibility. We note that the interpretation is given with respect to type derivations rather than type judgements.

We call this model the *interactive semantic model* for $\Lambda^!_S$. The interactive semantic model gives rise to another semantically obtained family of metrics. For terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we define $d^{\mathrm{int}}_{\Gamma,\tau}(M, N) \in \mathbb{R}^{\infty}_{\geq 0}$ by

$$d^{\mathrm{int}}_{\Gamma,\tau}(M, N) = d([\![M]\!]^{\mathrm{int}}, [\![N]\!]^{\mathrm{int}}).$$

We prove adequacy of the interactive semantic model, which will be used to prove $d^{\mathrm{obs}} \leq d^{\mathrm{int}}$. Below, for $\vdash M : \tau$, we write $[\![M]\!] \Downarrow$ when there is no $f : I \to [\![\tau]\!]$ such that $[\![M]\!] = f \otimes \bot_{(K,K)}$ where $\bot_X$ denotes the least element of $\mathbf{Int}(\mathbf{MetCppo})(I, X)$.

▶ **Theorem 49.** *Let $\vdash M : \tau$ be a term in $\Lambda^!_S$.*

- *If $M \hookrightarrow V$, then there is a derivation of $\vdash V : \tau$ such that $[\![M]\!] = [\![V]\!]$.*
- *If $[\![M]\!] \Downarrow$, then there is a value V such that $M \hookrightarrow V$.*

Proof. We can show the first claim by induction on the derivation of $M \hookrightarrow V$ using pointwise naturality of morphisms in Section 13.1.1, and we omit the detail. We prove the second claim by means of logical relations. For a type τ, we define a binary relation

$$P_\tau \subseteq \mathbf{Int}(\mathbf{MetCppo})(I, [\![\tau]\!]) \times \mathbf{Value}^!(\tau)$$

by

$$\begin{aligned}
P_{\mathbf{R}} &= \{(\llbracket \overline{a} \rrbracket, \overline{a}) \mid a \in \mathbb{R}\},\\
P_{\mathbf{I}} &= \{(\llbracket * \rrbracket, *)\},\\
P_{\tau\otimes\sigma} &= \{(f \otimes g, V \otimes U) \mid (f, V) \in P_\tau \text{ and } (g, U) \in P_\sigma\},\\
P_{\tau\multimap\sigma} &= \{(f, V) \mid \forall (g, U) \in P_\tau,\ (f \bullet g, V\,U) \in \overline{P}_\sigma\},\\
P_{!_n\tau} &= \{(n \cdot f, !V) \mid (f, V) \in P_\tau\}
\end{aligned}$$

where

$$\overline{P}_\tau = \{(\eta \circ f, M) \mid M \hookrightarrow V \text{ and } (f, V) \in P_\tau\} \\ \cup \{(f \otimes \bot_{(K,K)}, M) \mid f\colon I \to \llbracket \tau \rrbracket \text{ and } M \in \mathbf{Term}^!(\tau)\}$$

and $f \bullet g$ is given by

$$I \xrightarrow{f\otimes g} (\llbracket \tau \rrbracket \multimap T\llbracket \sigma \rrbracket) \otimes \llbracket \tau \rrbracket \xrightarrow{\text{eval}} T\llbracket \sigma \rrbracket.$$

By the definition of P_τ, we can show that P_τ and $\overline{P}_\tau$ are closed under taking least upper bounds of the first component: for all $(x_1, M), (x_2, M), \ldots \in P_\tau$, if $x_1 \leq x_2 \leq \cdots$, then we have $\left(\bigvee_{n\in\mathbb{N}} x_n, M\right) \in P_\tau$. We show basic lemma for P_τ: for any $\Gamma = (x :_{n_1} \sigma, \ldots, y :_{n_k} \rho)$, any $\Gamma \vdash M : \tau$ and any $(v, V) \in P_\sigma, \ldots, (u, U) \in P_\rho$, we have

$$(\llbracket M \rrbracket \circ ((n_1 \cdot v) \otimes \cdots \otimes (n_k \cdot u)), M[V/x, \ldots, U/y]) \in \overline{P}_\tau.$$

We only check the case for **fix**. The other cases are not difficult to check. What we check is: for environments $\Gamma = (x_1 :_{k_1} \rho_1, \ldots, x_n :_{k_n} \rho_n)$ and $\Delta = (x_1 :_{k_1'} \rho_1, \ldots, x_n :_{k_n'} \rho_n)$ such that $\infty{\cdot}\Gamma \leq \Delta$ and for a term $\Delta \vdash \mathbf{fix}_{\tau,\sigma}(f, x, M) : \tau \multimap \sigma$, given $(v_1, V_1) \in P_{\rho_1}, \ldots, (v_n, V_n) \in P_{\rho_n}$, we have

$$(\llbracket \mathbf{fix}_{\tau,\sigma}(f, x, M) \rrbracket \circ ((k_1 \cdot v_n) \otimes \cdots \otimes (k_n \cdot v_n)), \mathbf{fix}_{\tau,\sigma}(f, x, M)[V_1/x_1, \ldots, V_n/x_n]) \in P_{\tau\multimap\sigma}.$$

Let $\varphi\colon \mathbf{Int}(\mathbf{MetCppo})(\infty \cdot \llbracket \Gamma \rrbracket, \tau \multimap \sigma) \to \mathbf{Int}(\mathbf{MetCppo})(\infty \cdot \llbracket \Gamma \rrbracket, \tau \multimap \sigma)$ be the function given in Figure 12. Then, by induction on m, we can show that

$$\left(I \xrightarrow{(k_1' \cdot v_1)\otimes\cdots\otimes(k_n' \cdot v_n)} \llbracket \Delta \rrbracket \xrightarrow{d} \llbracket \infty \cdot \Gamma \rrbracket \xrightarrow{\varphi^m} \llbracket \tau \multimap \sigma \rrbracket, \mathbf{fix}_{\tau,\sigma}(f, x, M)[V_1/x_1, \ldots, V_n/x_n]\right)$$

is an element of $P_{\tau\multimap\sigma}$. Since $P_{\tau\multimap\tau'}$ is closed under least upper bounds on the first component, we obtain the claim. ◀

▶ **Theorem 50.** *For any pair of terms $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$, we have $d^{\text{obs}}_{\Gamma,\tau}(M, N) \leq d^{\text{int}}(M, N)$.*

Proof. We can prove the statement in the same way with Theorem 48. ◀

14 Conclusion

In this paper we study quantitative reasoning about linearly typed higher-order programs. We introduce a notion of admissibility for families of metrics on a purely linear programming language Λ_S, and among them, we investigate five notions of program metrics and how these are related, namely the logical metric, observational metric, equational metric, denotational

$$
\begin{aligned}
&[\![\Gamma \vdash \overline{a} : \mathbf{R}]\!] = [\![\Gamma]\!] \xrightarrow{w} I \xrightarrow{\lfloor a \rfloor} (R, I) \xrightarrow{\eta} T(R, I) \\
&[\![\Gamma \vdash * : \mathbf{I}]\!] = [\![\Gamma]\!] \xrightarrow{w} I \xrightarrow{\eta} TI \\
&[\![\Gamma_1 + \Gamma_2 \vdash \overline{f}(M_1, M_2) : \mathbf{R}]\!] = \\
&\quad [\![\Gamma_1 + \Gamma_2]\!] \xrightarrow{c} [\![\Gamma_1]\!] \otimes [\![\Gamma_2]\!] \xrightarrow{[\![M_1]\!] \otimes [\![M_2]\!]} T[\![\mathbf{R}]\!] \otimes T[\![\mathbf{R}]\!] \xrightarrow{\mathrm{dstr}} T([\![\mathbf{R}]\!] \otimes [\![\mathbf{R}]\!]) \xrightarrow{T\lfloor f \rfloor} T[\![\mathbf{R}]\!] \\
&[\![x :_n \tau \vdash x : \tau]\!] = n \cdot [\![\tau]\!] \xrightarrow{d} [\![\tau]\!] \xrightarrow{\eta} T[\![\tau]\!] \\
&[\![\Delta \vdash !M : !_n \tau]\!] = [\![\Delta]\!] \xrightarrow{\tilde{d}} n \cdot [\![\Gamma]\!] \xrightarrow{n \cdot [\![M]\!]} n \cdot T[\![\tau]\!] \xrightarrow{\xi} T[\![!_n \tau]\!] \\
&[\![\Delta \vdash \mathbf{fix}_{\tau,\sigma}(f, x, M) : \tau \multimap \sigma]\!] = [\![\Delta]\!] \xrightarrow{\tilde{d}} \infty \cdot [\![\Gamma]\!] \xrightarrow{\text{the least fixed point of } \varphi} \tau \multimap \sigma \\
&[\![\Gamma + \Delta \vdash M + N : \mathbf{R}]\!] = \\
&\quad [\![\Gamma + \Delta]\!] \xrightarrow{c} [\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{[\![M]\!] \otimes [\![N]\!]} T[\![\mathbf{R}]\!] \otimes T[\![\mathbf{R}]\!] \xrightarrow{\mathrm{dstr}} T([\![\mathbf{R}]\!] \otimes [\![\mathbf{R}]\!]) \xrightarrow{T(+)} T[\![\mathbf{R}]\!] \\
&[\![\Delta \vdash \overline{a} \cdot M : \mathbf{R}]\!] = [\![\Delta]\!] \xrightarrow{\tilde{d}} [\![\Gamma]\!] \xrightarrow{[\![M]\!]} n \cdot [\![\mathbf{R}]\!] \xrightarrow{\mathrm{mult}_{n,a}} [\![\mathbf{R}]\!] \xrightarrow{\eta} T[\![\mathbf{R}]\!] \\
&[\![\Gamma \vdash \lambda x : \sigma.\, M : \sigma \multimap \tau]\!] = (\text{the currying of } [\![\Gamma]\!] \otimes [\![\sigma]\!] \xrightarrow{[\![M]\!]} [\![\tau]\!]) \xrightarrow{\eta} T[\![\sigma \multimap \tau]\!] \\
&[\![\Gamma + \Delta \vdash M\, N : \tau]\!] = [\![\Gamma + \Delta]\!] \xrightarrow{c} [\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{[\![M]\!] \otimes [\![N]\!]} [\![\tau \multimap \sigma]\!] \otimes [\![\tau]\!] \xrightarrow{\mathrm{eval}} T[\![\sigma]\!] \\
&[\![\Gamma + \Delta \vdash M \otimes N : \tau \otimes \sigma]\!] = [\![\Gamma + \Delta]\!] \xrightarrow{c} [\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{[\![M]\!] \otimes [\![N]\!]} T[\![\tau]\!] \otimes T[\![\sigma]\!] \xrightarrow{\mathrm{dstr}} T[\![\tau \otimes \sigma]\!] \\
&[\![\Delta + \Xi \vdash \mathbf{let}\ *\ \mathbf{be}\ M\ \mathbf{in}\ N : \tau]\!] = [\![\Delta + \Xi]\!] \xrightarrow{c} [\![\Xi]\!] \otimes [\![\Delta]\!] \xrightarrow{\tilde{d}} [\![\Xi]\!] \otimes n \cdot [\![\Gamma]\!] \xrightarrow{[\![N]\!] \otimes ((n \cdot [\![M]\!]); \xi)} \\
&\quad T[\![\tau]\!] \otimes TI \xrightarrow{\mathrm{dstr}} T[\![\tau]\!] \\
&[\![\Delta + \Xi \vdash \mathbf{let}\ x \otimes y\ \mathbf{be}\ M\ \mathbf{in}\ N : \tau]\!] = [\![\Delta + \Xi]\!] \xrightarrow{c} [\![\Xi]\!] \otimes [\![\Delta]\!] \xrightarrow{\tilde{d}} [\![\Xi]\!] \otimes n \cdot [\![\Gamma]\!] \xrightarrow{[\![\Xi]\!] \otimes ((n \cdot [\![M]\!]); \xi)} \\
&\quad [\![\Xi]\!] \otimes T(n \cdot ([\![\sigma_1]\!] \otimes [\![\sigma_2]\!])) \xrightarrow{\cong} [\![\Xi]\!] \otimes T(n \cdot [\![\sigma_1]\!] \otimes n \cdot [\![\sigma_2]\!]) \xrightarrow{\mathrm{str}; [\![N]\!]; \mu} T[\![\tau]\!] \\
&[\![\Delta + \Xi \vdash \mathbf{let}\ !x\ \mathbf{be}\ M\ \mathbf{in}\ N : \tau]\!] = [\![\Delta + \Xi]\!] \xrightarrow{c} [\![\Xi]\!] \otimes [\![\Delta]\!] \xrightarrow{\tilde{d}} [\![\Xi]\!] \otimes n \cdot [\![\Gamma]\!] \xrightarrow{[\![\Xi]\!] \otimes ((n \cdot [\![M]\!]); \xi)} \\
&\quad [\![\Xi]\!] \otimes T(n \cdot m \cdot [\![\sigma]\!]) \xrightarrow{\cong} [\![\Xi]\!] \otimes T(nm \cdot [\![\sigma]\!]) \xrightarrow{\mathrm{str}; [\![N]\!]; \mu} T[\![\tau]\!]
\end{aligned}
$$

where $\varphi\colon \mathbf{Int}(\mathbf{MetCppo})(\infty \cdot [\![\Gamma]\!], \tau \multimap \sigma) \to \mathbf{Int}(\mathbf{MetCppo})(\infty \cdot [\![\Gamma]\!], \tau \multimap \sigma)$ is given by

$$
\begin{aligned}
\varphi(f) = \infty \cdot [\![\Gamma]\!] &\xrightarrow{c} \infty \cdot [\![\Gamma]\!] \otimes \infty \cdot [\![\Gamma]\!] \xrightarrow{d \otimes \delta} [\![\Gamma]\!] \otimes \infty \cdot \infty \cdot [\![\Gamma]\!] \\
&\xrightarrow{[\![\Gamma]\!] \otimes \infty \cdot f} [\![\Gamma]\!] \otimes \infty \cdot (\tau \multimap \sigma) \xrightarrow{\text{the currying of } [\![M]\!]} \tau \multimap \sigma
\end{aligned}
$$

Figure 12 Interpretation of Terms

metric and interactive metric. Some of our results can be seen as quantitative analogues of well-known results about program equivalences: the observational metric is less discriminating than or equal to semantic metrics, and non-definable functionals in the semantics are the source of inclusions. Existence of fully abstract semantics for Λ_S with respect to the observational metric is left open. Our study reveals the intrinsic difficulty of comparing denotational models with interactive semantic models obtained by applying the Int-construction. Indeed, their relationship is not trivial already at the level of program equivalences. It follows from [16] that there is a symmetric monoidal coreflection between **Int**(**MetCppo**) and **MetCppo**. This is a strong connection between these models. However, we do not know whether this categorical structure sheds light on their relationship at the level of higher-order programs.

Some of our results can be extended to a fragment of Fuzz where grading is restricted to extended natural numbers. Providing a quantitative equational theory and an interactive metric for full Fuzz is another very interesting topic for future work. There are some notions of metric that we have not taken into account in this paper. In [13], Gavazzo gives coinductively defined metrics for an extension of Fuzz with algebraic effects and recursive types, which we do not consider here. The so-called observational quotient [17] is a way to construct less discriminating program metrics from fine-grained ones. A thorough comparison of these notions of program distance with the ones we introduce here is another intriguing problem on which we plan to work in the future.

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